Cambridge Assessment
International Education

Specimen Paper Answers

Paper 3

Cambridge International AS & A Level
Further Mathematics 9231

For examination from 2020

Version 1

Cambridge Pathway

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Introduction

The main aim of this booklet is to exemplify standards for those teaching Cambridge International AS & A Level Further Mathematics 9231, and to show examples of model answers to the 2020 Specimen Paper 3. Paper 3 assesses the syllabus content for Further Mechanics. We have provided answers for each question in the specimen paper, along with examiner comments explaining where and why marks were awarded. Candidates need to demonstrate the appropriate techniques, as well as applying their knowledge when solving problems.

You will need to use the mark scheme alongside this document. This can be found on the School Support Hub (www.cambridgeinternational.org/support) on the 'Syllabus materials' tab – scroll down to the bottom of the page where the Specimen Paper materials are.

Individual examination questions may involve ideas and methods from more than one section of the syllabus content for that component. The main focus of examination questions will be the AS & A Level Further Mathematics syllabus content. However, candidates may need to make use of prior knowledge and mathematical techniques from previous study, as listed in the introduction to section 3 of the syllabus.

There are five to seven structured questions in Paper 3; candidates must answer **all** questions. Questions are of varied lengths and often contain several parts, labelled (a), (b), (c), which may have sub-parts (i), (ii), (iii), as needed. Some questions might require candidates to sketch graphs or diagrams, or draw accurate graphs.

Candidates are expected to answer directly on the question paper. All working should be shown neatly and clearly in the spaces provided for each question. New questions often start on a fresh page, so more answer space may be provided than is needed. If additional space is required, candidates should use the lined page at the end of the question paper, where the question number or numbers must be clearly shown.

Past exam resources and other teacher support materials are available on the School Support Hub (www.cambridgeinternational.org/support).

Assessment overview

There are three routes for Cambridge International AS & A Level Further Mathematics. Candidates may combine components as shown below.

Route 1 **AS Level only** (Candidates take the AS components in the same series)	**Paper 1** **Further Pure Mathematics 1**	**Paper 2** **Further Pure Mathematics 2**	**Paper 3** **Further Mechanics**	**Paper 4** **Further Probability & Statistics**
Either	✓	Not available for AS Level	✓	
Or	✓			✓

Route 2 **A Level (staged over two years)**	**Paper 1** **Further Pure Mathematics 1**	**Paper 2** **Further Pure Mathematics 2**	**Paper 3** **Further Mechanics**	**Paper 4** **Further Probability & Statistics**
Either Year 1 AS Level	✓		✓	
Year 2 Complete the A Level		✓		✓
Or Year 1 AS Level	✓			✓
Year 2 Complete the A Level		✓	✓	

Route 3 **A Level** (Candidates take the A Level components in the same series)	**Paper 1** **Further Pure Mathematics 1**	**Paper 2** **Further Pure Mathematics 2**	**Paper 3** **Further Mechanics**	**Paper 4** **Further Probability & Statistics**
Year 2 full A Level	✓	✓	✓	✓

Paper 3 – Further Mechanics

- Written examination, 1 hour 30 minutes, 50 marks
- 5 to 7 structured questions based on the Further Mechanics subject content
- Candidates answer all questions
- Externally assessed by Cambridge International
- 40% of the AS Level
- 20% of the A Level

Offered as part of AS Level or A Level

Assessment objectives

The assessment objectives (AOs) are the same for all papers:

AO1 Knowledge and understanding

- Show understanding of relevant mathematical concepts, terminology and notation
- Recall accurately and use appropriate mathematical manipulative techniques

AO2 Application and communication

- Recognise the appropriate mathematical procedure for a given situation
- Apply appropriate combinations of mathematical skills and techniques in solving problems
- Present relevant mathematical work, and communicate corresponding conclusions, in a clear and logical way

Weightings for assessment objectives

The approximate weightings (± 5%) allocated to each of the AOs are summarised below.

Assessment objectives as an approximate percentage of each component

Assessment objective	Weighting in components %			
	Paper 1	Paper 2	Paper 3	Paper 4
AO1 Knowledge and understanding	45	45	45	45
AO2 Application and communication	55	55	55	55

Assessment objectives as an approximate percentage of each qualification

Assessment objective	Weighting in AS Level %	Weighting in A Level %
AO1 Knowledge and understanding	45	45
AO2 Application and communication	55	55

Question 1

1 A child's toy consists of a uniform solid circular cone, of vertical height $3r$ and radius r, and a uniform solid hemisphere of radius r. The circular bases of the cone and the hemisphere are joined together so that they coincide. The cone and the hemisphere are made of the same material.

Show that the centre of mass of the toy is at a distance $\frac{27}{10}r$ from the vertex of the cone. [4]

Volumes: (1)

Cone $\frac{1}{3}\pi r^2 \times 3r = \pi r^3$

Hemisphere: $\frac{2}{3}\pi r^3$

Toy: $\pi r^3 + \frac{2}{3}\pi r^3 = \frac{5\pi r^3}{3}$

Centre of mass from vertex of cone: (2)

Cone: $\frac{3}{4} \times 3r = 9\frac{r}{4}$

Hemisphere: $3r + 3\frac{r}{8} = 27\frac{r}{8}$

Toy: $\bar{x}$

(response continues on next page)…

Examiner comment

Finding or stating a correct expression for the distance from the cone's vertex of the centre of mass of either the cone or the hemisphere gains B1.
A reasonable attempt to equate the sum of the moments of the cone and hemisphere about the vertex to that of the toy gains M1. Two accuracy marks are awarded for either solving this moment equation to show that the centre of mass of the toy is at the given distance from the vertex, or equivalently verifying that the given distance satisfies the moment equation. Equating moments about any other point on the axis of symmetry can also lead to the given solution and gain full marks.

(1) Candidates may use prior knowledge of volumes from IGCSE or O Level or may find them as volumes of revolution as in 9709 Paper 1. While no mark is awarded specifically for these volumes, incorrect expressions may well adversely affect the award of the final two accuracy marks.

Denoting density by ρ, equate moments about vertex: 3

$$5\pi\rho\frac{r^3}{3}\times\bar{x}=\pi\rho r^3\times 9\frac{r}{4}+\frac{2}{3}\pi\rho r^3\times 27\frac{r}{8}$$ 4

$$\bar{x}=\left(\frac{3}{5}\right)\left(9\frac{r}{4}+9\frac{r}{4}\right)=27\frac{r}{10}$$

Examiner comment

2 The positions of the centres of mass of a uniform solid cone and a uniform solid hemisphere may be found from the *List of formulae and statistical tables (MF19)*. The meaning of $\bar{x}$ is sufficiently obvious here for its definition to be reasonably omitted, but it is good practice to define such symbols when they are first introduced.

3 Since the cone and hemisphere are here made of the same material, their densities are the same and so ρ could reasonably be omitted entirely. It is however, good practice to include the density when finding a moment from a volume, since some situations may involve components with unequal densities. It is also good practice to indicate the basis of an equation before writing it down.

4 Some candidates may choose to simply write down the moment equation directly, without first finding explicit expressions for the volumes and distances required. Such a concise response can gain full marks provided all the terms are correct and their derivation is sufficiently clear, as is the case in the equation indicated by 4 above, even though such candidates may write down only three lines or so in total. A more simplified equation is unlikely to be sufficient by itself, so just writing down, for example, $\frac{5\pi r^3}{3}\times\bar{x}=\frac{9\pi r^4}{4}+\frac{9\pi r^4}{4}$ and solving for $\bar{x}$ would not justify full marks.

Candidates may choose to draw a diagram to better visualise the structure of the toy, but this is not required.

Question 2

2 A light elastic string has natural length a and modulus of elasticity $24mg$. One end of the string is attached to a fixed point A. The other end of the string is attached to a particle of mass $2m$.

(a) Find, in terms of a, the extension of the string when the particle hangs freely in equilibrium below A. [2]

Denoting by x the required extension of the string in equilibrium, (1)

equate vertical forces on the particle using Hooke's Law: (2)

$$24mg\frac{x}{a} = 2mg$$

$$x = \frac{a}{12}$$

Examiner comment

A reasonable attempt to equate the upward force on the particle due to the tension (found using Hooke's Law) to the weight gains the method mark. The correct solution of the equation gains an accuracy mark. Candidates may choose to draw a diagram (in either part) to better visualise the forces and distances involved, but this is not required.

(1) While the meaning of *x* may be obvious here, it is good practice to define it explicitly. It is of course acceptable to find some other distance, such as that of the particle below *A* in equilibrium, and hence the required extension. A similar observation applies to *d* in part **(b)**.

(2) The origin of the equation is fairly obvious here, and similarly in part **(b)**, but it is good practice to explain it.

(b) The particle is released from rest at *A*.

Find, in terms of *a*, the distance of the particle below *A* when it first comes to instantaneous rest. [6]

Denoting by d the required distance of the particle below A, use conservation of energy:

$$2mg \times d = \frac{1}{2}24mg\frac{(d-a)^2}{a}$$ (3)

$$6d^2 - 13ad + 6a^2 = 0 = (3d-2a)(2d-3a)$$

rejecting $d = \frac{2a}{3} < a$, $d = \frac{3a}{2}$ (4)

An alternative acceptable response would be to first note that the particle will fall under gravity to the point *B* where the string first becomes taut, at which point its speed v_B will satisfy $v_B^2 = 2ga$. Since its subsequent downward motion will be simple harmonic, v_B^2 will also equal $\omega^2(x_0^2 - x^2)$ with $\omega^2 = \frac{12g}{a}$ and $x = \frac{a}{12}$ to give the amplitude $x_0 = \frac{5a}{12}$ and hence the required distance $d = a + \frac{a}{12} + x_0 = \frac{3a}{2}$.

Examiner comment

A reasonable attempt to equate the loss in gravitational potential energy to the gain in elastic potential energy gains M1, with B1 awarded for formulating each of the two expressions correctly. A second M1 is gained for a reasonable attempt to solve this energy equation, with two accuracy marks for finding both possible solutions correctly and rejecting the inadmissible one.

(3) Since the particle is at rest both at *A* and at the lowest point of the motion (*C* say), it is unnecessary to introduce kinetic energy. Some candidates may choose however, to consider the changes in energy between *A* and some intermediate point such as when the string first becomes taut, and then between this point and *C*. In this case a correct solution would require consideration of kinetic energy, thus introducing an unnecessary complication though full marks could still be gained.

(4) Although some candidates may regard it as self-evident that $d = \frac{2a}{3}$ is inadmissible since it corresponds to a point at which the string is slack, it should be rejected explicitly.

Question 3

3 A particle P of mass m kg falls from rest under gravity. There is a resistive force of magnitude mkv^2 N, where v m s^{-1} is the speed of P after it has fallen a distance x m and k is a positive constant.

(a) By solving an appropriate differential equation, show that $v^2 = \frac{8}{k}(1 - e^{-2kt})$ [7]

Use Newton's 2nd Law of motion in vertical direction: (1)

$$mg - mkv^2 = mv\frac{dv}{dx}$$ (2)

$$\int dx = \int \left\{\frac{v}{(g-kv^2)}\right\} dv$$

$$x = \left(-\frac{1}{2k}\right)\ln(g-kv^2) + c$$

When $x = 0,\ v = 0$: $c = \left(\frac{1}{2k}\right)\ln g$ so $x = \left(\frac{1}{2k}\right)\ln\left\{\frac{g}{(g-kv^2)}\right\}$

$$e^{2kx} = \frac{g}{(g-kv^2)}$$

$$g - kv^2 = ge^{-2kx}$$

$$v^2 = \left(\frac{g}{k}\right)(1-e^{-2kx})$$ (3)

Examiner comment

Using Newton's Law to formulate a correct differential equation relating the variables v and x gains B1. M1 is awarded for a reasonable attempt to separate variables and integrate, with an accuracy mark (A1) if this is performed correctly. A further M1 and A1 are gained by applying the given initial conditions, with the final M1 and A1 being awarded for rearranging into the given form.

(1) It is good practice to state the origin of the equation, particularly in a question requiring a given result to be shown, as here.

(2) The mass m is common to all terms of the equation and could therefore be omitted, but it is good practice to include it initially. The acceleration is expressed as $v\frac{dv}{dx}$ rather than $\frac{d^2x}{dt^2}$ so that the variables can be separated and the resulting terms integrated.

(3) The final result is given, so sufficient working must be shown in order to justify the award of full marks. Recall that communication, including the expression of mathematical proof and reasoning in such a way that others can follow it, is one of the key concepts for Further Mathematics.

It is now given that $k = 0.01$. The speed of P when x becomes large approaches V m s^{-1}.

(b)(i) Find V correct to 2 decimal places. [1]

As x increases, $v^2 \to \frac{g}{k}$ so $V = \sqrt{(1000)} = 31.62$

Examiner comment

The single mark is awarded for a numerical answer correct to 2 decimal places. Note that this is a departure from the default rubric requirement of 3 significant figures. Units are not required since it is the numerical value of *V* that must be found. Candidates are instructed to use 10 ms^{-2} for *g* in this question.

(b)(ii) Hence find how far P has fallen when its speed is $\frac{1}{2}V$ms^{-1}. [2]

When $v = \frac{1}{2}V$, $v^2 = \frac{1}{4}V^2 = \frac{1}{4}\frac{g}{k}$ so

$$x = \left(\frac{1}{2k}\right)\ln\left\{\frac{g}{\left(g - \frac{1}{4}g\right)}\right\} = 50\ \ln\frac{4}{3} \text{ and required distance is } 14.4 \text{ m}$$

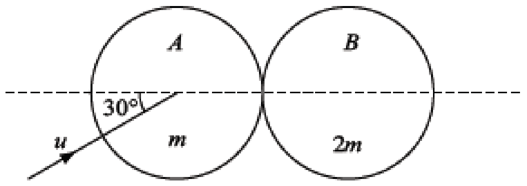
Examiner comment

M1 is awarded for a reasonable attempt to find *x* at the specified point, with an accuracy mark for doing so correctly. Candidates' working does not need to take the form shown here, but they are advised to show sufficient working to justify M1 even if an error occurs and their final result is incorrect.

In accordance with the rubric, the final result should be accurate to at least 3 significant figures, and inclusion of the unit (metres) is appropriate here although its omission would not usually be penalised.

Question 4

4



Two uniform smooth spheres A and B of equal radii have masses m and $2m$ respectively. Sphere B is at rest on a smooth horizontal surface. Sphere A is moving on the surface with speed u at an angle of 30° to the line of centres of A and B when it collides with B (see diagram). The coefficient of restitution between the spheres is e.

(a) Show that the speed of B after the collision is $\frac{\sqrt{3}}{6}u(1+e)$ and find the speed of A after the collision. [6]

Denoting the centres of A, B at the instant of collision by O_A, O_B

and the speeds of A, B after the collision in the direction O_AO_B by v_A, v_B, (1)

(2) use conservation of momentum:

$$m\,v_A + 2m\,v_B = mu\cos 30°$$

Use Newton's restitution equation:

$$v_A - v_B = -eu\cos 30°$$

Combine:

$$3\,v_B = u(1+e)\cos 30°,\quad v_B = \left(\frac{\sqrt{3}}{6}\right)u(1+e)$$

$$v_A = \left(\frac{\sqrt{3}}{6}\right)u(1-2e)$$

Find speed of A:

$$\sqrt{\left\{(u\sin 30°)^2 + V_A{}^2\right\}} = u\sqrt{\left\{\frac{1}{4} + \frac{(1-2e)^2}{12}\right\}} = u\sqrt{\left\{\frac{(1-e+e^2)}{3}\right\}}$$

Examiner comment

M1 is awarded for a reasonable attempt to use conservation of momentum in the direction O_AO_B and another M1 for a reasonable attempt to use Newton's restitution equation. Combining these two equations to verify the given speed of B and to find the speed of A in the direction O_AO_B earns A1 for each. A third M1 is awarded for combining the latter speed with the unchanged speed of A in the direction perpendicular to O_AO_B, with A1 for the correct final result.

(1) While the meaning of v_A and v_B may be sufficiently clear from the working, it is good practice to define them either in words or by means of a diagram. Some candidates may choose to represent the overall speed of A after the collision as V_A at an angle of θ to O_AO_B, so that $V_A\cos\theta = v_A$ and $V_A\sin\theta = u\sin 30°$, which is of course acceptable.

(2) It is good practice to state the origin of the two equations.

(b) Given that $e = \frac{1}{3}$, find the loss of kinetic energy as a result of the collision. [3]

When $e = \frac{1}{3}$, loss of K.E. is

$$\frac{1}{2}mu^2 - \frac{1}{2}mu^2\frac{\left(1-\frac{1}{3}+\frac{1}{3^2}\right)}{3} - \frac{1}{2}2mu^2\left(\frac{1}{12}\right)\left(1+\frac{1}{3}\right)^2$$

$$= \frac{1}{2}mu^2\left(1-\frac{7}{27}-\frac{8}{27}\right) = \left(\frac{2}{9}\right)mu^2$$

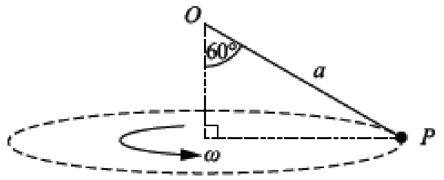
Examiner comment

A method mark is awarded for a reasonable attempt to subtract the kinetic energies of *A* and *B* after the collision from that of *A* before the collision, with $e = \frac{1}{3}$. One A1 is awarded for the resulting (unsimplified) expression for the required loss of kinetic energy, following through on the speed of *A* found in the first part of the question. A second A1 is awarded for a correct final answer.

Question 5

5 A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O.

(a)



The particle P moves in a horizontal circle with a constant angular speed ω with the string inclined at 60° to the downward vertical through O (see diagram).

Show that $\omega^2 = \dfrac{2g}{a}$. [4]

Denoting the tension in the string by T,

equate the vertical forces on P:

$$T\cos 60° = mg$$

Use Newton's second law in a radial direction:

$$T\sin 60° = m(a\sin 60°)\omega^2$$

Combine:

$$\omega^2 = \frac{mg}{ma}\cos 60° = \frac{2g}{a}$$

Examiner comment

B1 is awarded for equating the vertical forces correctly. A reasonable attempt to use Newton's law radially earns M1, with A1 if the equation is correct. Another A1 is awarded for verifying the given answer correctly.

As in the previous questions, it is good practice to define any symbols introduced, here the tension T, and to indicate the origin of the equations.

(b) The particle now hangs at rest a distance a vertically below O. It is then projected horizontally so that it begins to move in a vertical circle with centre O. When the string makes an angle of 60° with the downward vertical through O, the angular speed of P is $\sqrt{\frac{2g}{a}}$. The string first goes slack when OP makes an angle θ with the upward vertical through O.

Find the value of $\cos\theta$. [6]

Denoting the angular speed of the particle when the string first goes slack by ω_1, (1)

use conservation of energy: (2)

$$\frac{1}{2}m(a\omega_1)^2 = \frac{1}{2}m \times 2ga - mga(\cos 60° + \cos\theta)$$

Use Newton's second law radially when string first goes slack, so that tension is zero:

$$mg\cos\theta = ma\omega_1^2$$

Combine:

$$a\omega_1^2 = g(1 - 2\cos\theta) = g\cos\theta,\ \cos\theta = \frac{1}{3}$$

Examiner comment

A reasonable attempt to use conservation of energy between the point where the string is at 60° to the downward vertical and the point where it is at θ to the upward vertical earns M1, with A1 being awarded if the equation is correct. Similarly, a reasonable attempt to use Newton's law radially at the latter point, possibly with tension included, earns M1. A1 is awarded if the tension is taken to be zero and the resulting equation is correct. The final M1 and A1 are awarded for combining these two equations to find the required value of cos θ.

(1) It is equally acceptable to work in terms of angular or linear speed. As in the previous questions, it is good practice to define any symbols introduced, here the speed when the string first goes slack. A different symbol ω_1 has been chosen instead of the ω used in part **(a)**, but the latter is of course acceptable provided no confusion occurs.

(2) It is also good practice to indicate the origin of the equations, and in this case to state explicitly that the tension is zero when the string becomes slack.

Question 6

6 A particle P is projected with speed u at an angle α above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a subsequent time t are denoted by x and y respectively.

(a) Derive the equation of the trajectory of P in the form

$$y = x\tan a - \frac{gx^2}{2u^2}\sec^2 a.$$

[3]

Find x and y by considering horizontal and vertical motion, respectively:

$$x = (u\cos\alpha)t, \quad y = (u\sin\alpha)t - \frac{1}{2}gt^2$$

Combine:

$$y = \frac{(u\sin\alpha)x}{(u\cos\alpha)} - \frac{1}{2}g\left\{\frac{x}{(u\cos\alpha)}\right\}^2$$

$$= x\tan\alpha - \left(\frac{gx^2}{2u^2}\right)\sec^2\alpha$$

Examiner comment

B1 is awarded for correctly expressing both x and y in terms of u, α, g and t. Combining these two equations to eliminate t and hence verify the given trajectory equation, then earns M1 A1.

Since the question requires that the trajectory equation be derived, it is of course not sufficient to simply quote it from the *List of formulae and statistical tables (MF19)*.

(b) The greatest height of P above the plane is denoted by H. When P is at a height of $\frac{3}{4}H$, it has travelled a horizontal distance d.

Given that $\tan\alpha = 2$, find, in terms of H, the two possible values of d. [6]

Find H when $\tan\alpha = 2$:

$$H = \frac{(u\sin\alpha)^2}{2g} = \frac{2u^2}{5g}$$

Put $x = d$, $y = \frac{3}{4}H$ in trajectory equation:

$$\frac{3}{4}H = 2d - \left(\frac{gd^2}{2u^2}\right) \times 5$$

Combine:

$$\frac{3}{4}H = \frac{2d - d^2}{H},\ 4d^2 - 8Hd + 3H^2 = 0 = (2d - H)(2d - 3H)$$

$$d = \frac{H}{2},\ 3\frac{H}{2}$$

Examiner comment

M1 is awarded for a reasonable attempt to find or state an expression for the greatest height H. A1 is awarded for finding H correctly when $\tan\alpha = 2$. A further M1 is awarded for substituting the given values of x, y in the trajectory equation, and a final M1 for a reasonable attempt to eliminate $\frac{u^2}{g}$ from these two equations. A resulting correct quadratic equation in d and H earns A1, with a final A1 for the required two possible values of d in terms of H.

Some candidates may recall the expression for the greatest height reached, while others will need to derive it.

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