Cambridge Assessment
International Education

Specimen Paper Answers

Paper 2

Cambridge International AS & A Level Further Mathematics 9231

For examination from 2020

Version 1

Cambridge Pathway

Contents

Introduction

The main aim of this booklet is to exemplify standards for those teaching Cambridge International AS & A Level Further Mathematics 9231, and to show examples of model answers to the 2020 Specimen Paper 2. Paper 2 assesses the syllabus content for Further Pure Mathematics 2. We have provided answers for each question in the specimen paper, along with examiner comments explaining where and why marks were awarded. Candidates need to demonstrate the appropriate techniques, as well as applying their knowledge when solving problems.

You will need to use the mark scheme alongside this document. This can be found on the School Support Hub (www.cambridgeinternational.org/support) on the ‘Syllabus materials’ tab – scroll down to the bottom of the page where the Specimen Paper materials are.

Individual examination questions may involve ideas and methods from more than one section of the syllabus content for that component. The main focus of examination questions will be the AS & A Level Further Mathematics syllabus content. However, candidates may need to make use of prior knowledge and mathematical techniques from previous study, as listed in the introduction to section 3 of the syllabus.

There are seven to nine structured questions in Paper 2; candidates must answer **all** questions. Questions are of varied lengths and often contain several parts, labelled (a), (b), (c), which may have sub-parts (i), (ii), (iii), as needed. Some questions might require candidates to sketch graphs or diagrams, or draw accurate graphs.

Candidates are expected to answer directly on the question paper. All working should be shown neatly and clearly in the spaces provided for each question. New questions often start on a fresh page, so more answer space may be provided than is needed. If additional space is required, candidates should use the lined page at the end of the question paper, where the question number or numbers must be clearly shown.

Past exam resources and other teacher support materials are available on the School Support Hub (www.cambridgeinternational.org/support).

Assessment overview

There are three routes for Cambridge International AS & A Level Further Mathematics. Candidates may combine components as shown below.

Route 1 **AS Level only** (Candidates take the AS components in the same series)	**Paper 1** **Further Pure Mathematics 1**	**Paper 2** **Further Pure Mathematics 2**	**Paper 3** **Further Mechanics**	**Paper 4** **Further Probability & Statistics**
Either	✓	Not available for AS Level	✓	
Or	✓			✓

Route 2 **A Level (staged over two years)**	**Paper 1** **Further Pure Mathematics 1**	**Paper 2** **Further Pure Mathematics 2**	**Paper 3** **Further Mechanics**	**Paper 4** **Further Probability & Statistics**
Either Year 1 AS Level	✓		✓	
Year 2 Complete the A Level		✓		✓
Or Year 1 AS Level	✓			✓
Year 2 Complete the A Level		✓	✓	

Route 3 **A Level** (Candidates take the A Level components in the same series)	**Paper 1** **Further Pure Mathematics 1**	**Paper 2** **Further Pure Mathematics 2**	**Paper 3** **Further Mechanics**	**Paper 4** **Further Probability & Statistics**
Year 2 full A Level	✓	✓	✓	✓

Paper 2 – Further Pure Mathematics 2

- Written examination, 2 hours, 75 marks
- 7 to 9 structured questions based on the Further Pure Mathematics 2 subject content
- Candidates answer all questions
- Externally assessed by Cambridge International
- 30% of the A Level only

This is compulsory for A Level.

Assessment objectives

The assessment objectives (AOs) are the same for all papers:

AO1 Knowledge and understanding

- Show understanding of relevant mathematical concepts, terminology and notation
- Recall accurately and use appropriate mathematical manipulative techniques

AO2 Application and communication

- Recognise the appropriate mathematical procedure for a given situation
- Apply appropriate combinations of mathematical skills and techniques in solving problems
- Present relevant mathematical work, and communicate corresponding conclusions, in a clear and logical way

Weightings for assessment objectives

The approximate weightings (± 5%) allocated to each of the AOs are summarised below.

Assessment objectives as an approximate percentage of each component

Assessment objective	Weighting in components %			
	Paper 1	**Paper 2**	**Paper 3**	**Paper 4**
AO1 Knowledge and understanding	45	45	45	45
AO2 Application and communication	55	55	55	55

Assessment objectives as an approximate percentage of each qualification

Assessment objective	Weighting in AS Level %	Weighting in A Level %
AO1 Knowledge and understanding	45	45
AO2 Application and communication	55	55

Question 1

1 Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 7 - 2t^2.$$ [6]

Auxiliary equation $m^2 + 4m + 4 = 0 \Rightarrow (m + 2)^2 = 0 \Rightarrow m = -2$ M1

So the complementary function is $x = (A + Bt)\,e^{-2t}$ A1

Particular integral $x = pt^2 + qt + r$ so $\dot{x} = 2pt + q$ and $\ddot{x} = 2p$ B1

Substituting in $2p + 4(2pt + q) + 4(pt^2 + qt + r) = 7 - 2t^2$

$4p = -2 \quad 8p + 4q = 0$ and $2p + 4q + 4r = 7$ M1

So $p = -\frac{1}{2}$, $q = -2p = 1$ and $4r = 7 - 4 \times 1 - 2 \times -\frac{1}{2} = 4$ so $r = 1$ A1

$x = (A + Bt)\,e^{-2t} + 1 + t + \frac{1}{2}t^2$ A1

Examiner comment

After gaining M1 for forming and solving the auxiliary equation, candidates need to remember the form of the complementary function when there is a real repeated root, and take care to use the correct variables for A1. They should set up the particular integral carefully and show $\dot{x}$ and $\ddot{x}$ for B1, before substituting to gain M1 and finding the coefficients for the second A1.

The general solution, combining the complementary function and particular integral should be written down, again being sure to use the correct variables for the final A mark. Marks are sometimes lost by candidates using the wrong variables in their answer.

Question 2

The mark scheme gives two alternative solutions for this question, so we have provided a model answer for both.

Method 1

2 Find the exact value of $\int_0^1 \frac{1}{\sqrt{3+4x-4x^2}}\,dx$. [6]

$3 + 4x - 4x^2 = 4 - (2x - 1)^2$ by completing the square M1

Let $2x - 1 = 2\sin\theta$ so $2\cos\theta = \sqrt{4-4\sin^2\theta}$ and $2\frac{dx}{d\theta} = 2\cos\theta\, d\theta$ M1

$$\int_0^1 \frac{1}{\sqrt{4-(2x-1)^2}}\,dx = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos\theta}{4-4\sin^2\theta}\,d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos\theta}{2\cos\theta}\,d\theta$$ M1

$$= \left[\frac{1}{2}\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$ A1

$$= \frac{\pi}{6}$$ M1 A1

Examiner comment

Completing the square is a standard technique for tackling various integrations with quadratic expressions and earns the first method mark. In this method, this leads into an integration by substitution, which gains another method mark, and the third method mark is for simplifying the expression. The first accuracy mark is awarded for the correct integration from full working. Using the correct limits (candidates may revert to the original variable) gives the final M1 and A1 if all is correct. No credit will be given for answers without working, or correct answers following incorrect working.

Method 2

2 Find the exact value of $\int_0^1 \frac{1}{\sqrt{3+4x-4x^2}}\,dx$. [6]

$3 + 4x - 4x^2 = 4 - (2x - 1)^2$ by completing the square M1

$\int_0^1 \frac{1}{\sqrt{4-(2x-1)^2}}\,dx = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-(x-\frac{1}{2})^2}}\,dx$ M1 $= \frac{1}{2}\sin^{-1}(x - \frac{1}{2})$ M1 A1

$= \frac{1}{2}(\sin^{-1}(0.5) - \sin^{-1}(-0.5))$ M1 $= \frac{\pi}{6}$ A1

Examiner comment

In this alternative method, the first method mark is awarded in the same way as for Method 1, the next M1 is for further manipulating the integrand into the form shown, so that it can be integrated using a result from the formula sheet for the next M1 and the first A1. Using the correct limits gains the final M1 and the final mark is an accuracy mark. No credit will be given for answers without working, or correct answers following incorrect working.

Question 3

3 Find the solution of the differential equation

$$x\frac{dy}{dx} + 3y = \frac{\sin x}{x}$$

for which $y = 0$ when $x = \frac{1}{2}\pi$. Give your answer in the form $y = f(x)$. [8]

Dividing through by x to make the coefficient of $\frac{dy}{dx}$ equal to one gives

$$\frac{dy}{dx} + y\frac{3}{x} = \frac{\sin x}{x^2} \quad \text{B1}$$

So the integrating factor is $e^{\int \frac{3}{x}dx}$ M1 $= e^{3\ln x} = x^3$ A1

Multiplying through gives $x^3\frac{dy}{dx} + 3yx^2 = x\sin x$

i.e. $\frac{d}{dx}(x^3y) = x\sin x$ M1 $\Rightarrow x^3y = \int x\sin x\,dx = -x\cos x - \int -\cos x\,dx$

$= -x\cos x + \sin x + c$ A1

Using $y = 0$, $x = \frac{\pi}{2}$ $\left(\frac{\pi}{2}\right)3 \times 0 = \left(-\frac{\pi}{2}\right) \times 0 + 1 + c$ so $c = -1$ M1

$$y = \frac{-\cos(x)}{x^2} + \frac{\sin(x)}{x^3} - \frac{1}{x^3} \quad \text{M1 A1}$$

Examiner comment

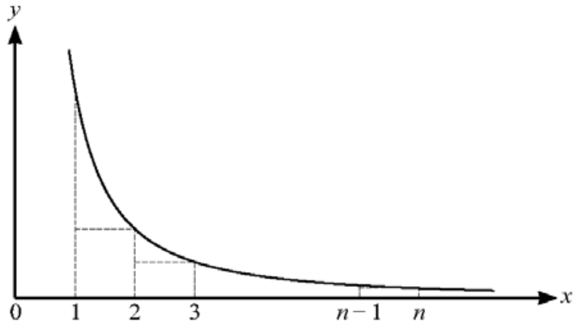
The first mark is for getting into the right form, then the method mark is for attempting to find the integrating factor, with an accuracy mark if it is correct. The integration is then set up, and the right hand side is integrated by parts for M1 and A1.

The value of the arbitrary constant is found by substituting the given values for M1, before the final rearrangement.

It is essential to show the working throughout this question so that it is clear that the integrations are being done, not taken from a calculator. The final answer must be given in the form required.

Question 4

4



The diagram shows the curve with equation $y = \frac{1}{x^2}$ for $x > 0$, together with a set of $(n-1)$ rectangles of unit width.

(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{1}{r^2} < \frac{2n-1}{n}.$$

[5]

Summing the areas of the given rectangles gives $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2}$ M1 A1

So $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2} < \int_1^n \frac{1}{x^2}dx$ M1 and $\int_1^n \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^n = 1 - \frac{1}{n}$ M1

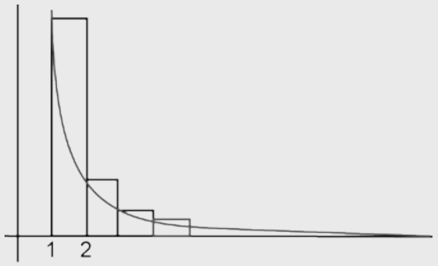
So $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2} < 1 + 1 - \frac{1}{n}$ so $\sum_{r=1}^{n} \frac{1}{r^2} < \frac{2n-1}{n}$ A1

Examiner comment

As the answer is given, it is very important to show detailed steps in the working. The first steps are to relate the areas of the rectangles to a summation and relate this summation to the area under the curve. It should then be made clear that 1 has to be added to both sides of the inequality for the final A1. It is also important to show the integration steps clearly.

(b) Use a similar method to find, in terms of n, a lower bound for $\sum_{r=1}^{n} \frac{1}{r^2}$. [3]

Method 1



Summing the rectangles that would sit above the curve including rectangle between n and $n+1$ gives $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}$ M1 $> \int_1^{n+1} \frac{1}{x^2}dx$ M1

$= \left[-\frac{1}{x}\right]_1^{n+1} = 1 - \frac{1}{n+1}$ A1

Examiner comment

A diagram or annotations on the given diagram would add clarity here, showing the rectangles needed above the curve. As in part **(a)** it is important to be exact about the summation and in the second case adjust the sum by adding $\frac{1}{n^2}$ to both sides of the inequality.

Method 2

Summing rectangles sitting above the curve up to rectangle between $n-1$ and n gives $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{(n-1)^2}$ M1 $> \int_1^{n} \frac{1}{x^2}dx$ M1 $= \left[-\frac{1}{x}\right]_1^{n} = 1 - \frac{1}{n}$

So adding last term to both sides gives $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2} > 1 - \frac{1}{n} + \frac{1}{n^2}$ A1

Examiner comment

A diagram or annotations on the given diagram would add clarity here showing the rectangles needed above the curve. As in part **(a)** it is important to be exact about the summation and in the second case adjust the sum by adding $\frac{1}{n^2}$ to both sides of the inequality.

Question 5

5 The curve C has parametric equations

$$x = e^t - 4t + 3, \quad y = 8e^{\frac{1}{2}t} \qquad \text{for } 0 \leqslant t \leqslant 2.$$

(a) Find, in terms of e, the length of C. [5]

$\frac{dx}{dt} = e^t - 4$, $\frac{dy}{dt} = 4e^{\frac{1}{2}t}$ B1 so $\dot{x}^2 + \dot{y}^2 = e^{2t} - 8e^t + 16 + 16e^t = e^{2t} + 8e^t + 16 = (e^t+4)^2$ M1 A1

Length of curve $= \int_0^2 \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = \int_0^2 (e^t+4)\, dt$ M1 $= \left[(e^t+4t)\right]_0^2 = e^2 + 8 - 1 = e^2 + 7$ A1

Examiner comment

Most candidates will spot the factorisation needed and gain the first M1 A1 having differentiated correctly (B1). They can then proceed to a simple integration to find the length of the curve (M1 A1). Answers do need to be left exact, and as always with integration, answers without working will not gain marks.

(b) Find, in terms of π and e, the area of the surface generated when C is rotated through 2π radians about the x-axis. [5]

Surface area $= 2\pi \int_0^2 y\sqrt{\dot{x}^2 + \dot{y}^2}\, dt = 2\pi \int_0^2 8e^{\frac{1}{2}t}(e^t+4)\, dt$ B1

$= 16\pi \int_0^2 e^{\frac{3}{2}t} + 4e^{\frac{1}{2}t}\, dt$ M1 $= 16\pi \left[\frac{2}{3}e^{\frac{3}{2}t} + 8e^{\frac{1}{2}t}\right]_0^2$ M1 A1

$= 16\pi \left(\frac{2}{3}e^3 + 8e - \frac{26}{3}\right)$ A1

Examiner comment

Most candidates recall the relevant formula (B1 follow through on (a)), though there can be confusion with the Cartesian form. Limits are not required for the first M1. The steps of the integration must be complete (M1 A1), and answers only will not be credited.

Question 6

6 (a) Using de Moivre's theorem, show that

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 = 10\tan^2\theta + 5\tan^4\theta}$$ [5]

By de Moivre's theorem, $\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$ B1

But $(\cos\theta + i\sin\theta)^5 = c^5 + 5\,c^4(is)^1 + 10\,c^3(is)^2 + 10\,c^2(is)^3 + 5\,c^1(is)^4 + (is)^5$ M1 A1

So, equating real parts $\cos 5\theta = c^5 + 10\,c^3(is)^2 + 5\,c^1(is)^4 = c^5 - 10\,c^3s^2 + 5\,c^1s^4$

Equating imaginary parts $\sin 5\theta = 5\,c^4s^1 - 10\,c^2s^3 + s^5$

So $\tan(5\theta) = \dfrac{5c^4s - 10c^2s^3 + s^5}{c^5 - 10c^3s^2 + 5cs^4}$ M1

And dividing numerator and denominator by c^5 gives

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$ A1

Examiner comment

Candidates need to justify each step as the answer is given. Solutions to these types of question are usually of a high standard, with accurate expansions and simplifications. Sometimes the last step, dividing by $\cos^5\theta$, is not made explicit and the final A1 would be lost.

Candidates need to be confident in using this application of de Moivre's theorem (as shown in the first line for B1), and not try to use the relationship between $\cos\theta$ and z and $\frac{1}{z}$ which would require considerably more manipulation. When a question specifies a particular method like this, this method must be used, so trigonometric manipulation alone will not be credited.

(b) Hence show that the equation $x^2 - 10x + 5 = 0$ has roots $\tan^2\left(\frac{1}{5}\pi\right)$ and $\tan^2\left(\frac{2}{5}\pi\right)$. [5]

Let $t = \tan\theta$, then $\tan 5\theta = \frac{5t - 10^3 + t^5}{1 - 10t^2 + 5t^4} = 0 \Rightarrow t(t^4 - 10t^2 + 5) = 0$

But $\tan 5\theta = 0 \Rightarrow 5\theta = 0, \pi, 2\pi, 3\pi, 4\pi$ so $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$ B1

$\theta = 0$ corresponds to $t = 0$ so roots of $t^4 - 10t^2 + 5 = 0$ are $\tan\frac{\pi}{5}$, $\tan\frac{2\pi}{5}$,

$\tan\frac{3\pi}{5}$, and $\tan\frac{4\pi}{5}$ B1

so $t^4 - 10t^2 + 5 = (t - \tan\frac{\pi}{5})(t - \tan\frac{2\pi}{5})(t - \tan\frac{3\pi}{5})(t - \tan\frac{4\pi}{5})$ M1

But $\tan\frac{4\pi}{5} = -\tan\frac{\pi}{5}$ and $\tan\frac{4\pi}{5} = -\tan\frac{2\pi}{5}$ M1

So equation becomes $(t^2 - \tan^2\frac{2\pi}{5})(t^2 - \tan^2\frac{2\pi}{5}) = 0$

And therefore letting $x = t^2$ the equation $x^2 - 10x + 5 = 0$ has roots $\tan^2\frac{\pi}{5}$ and

$\tan^2\frac{2\pi}{5}$. A1

Examiner comment

Strong candidates are able to make the connection between the equations and to select the relevant solutions for the two B marks. They then need to show clearly how the four solutions combine to give two distinct solutions to the quadratic in t^2 for the method marks – as the answer is given, the working must be detailed and clear.

Question 7

7 **(a)** Starting from the definition of tanh in terms of exponentials, prove that

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$ [3]

If $u = \tanh^{-1}x$ then $x = \tanh(u) = \frac{e^u - e^{-u}}{e^u + e^{-u}}$ writing in exponential form. M1

Then $(e^u + e^{-u})\tanh(u) = e^u - e^{-u}$ so $(e^{2u} + 1)x = (e^{2u} - 1)$ M1

(multiplying both sides by e^u)

Therefore $e^{2u}(1 - x) = 1 + x$ and so $e^{2u} = \frac{1+x}{1-x}$ and $u = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ A1

Examiner comment

This straightforward manipulation of exponentials follows directly from the definition of tanh(x), and needs to be communicated in detail, since the answer is given. The first method mark is for writing tanh(u) in exponential form, the second method mark is for the algebraic manipulation. The final A1 is dependent on seeing the steps in the final line (or equivalent) as the answer is given.

(b) Given that $y = \tanh^{-1}\left(\dfrac{1-x}{2+x}\right)$, show that $(2x+1)\dfrac{\mathrm{d}y}{\mathrm{d}x}+1=0$. [4]

Method 1

Using the result above, $y = \frac{1}{2}\ln\left(\dfrac{1+\frac{1-x}{2+x}}{1-\frac{1-x}{2+x}}\right)$ M1 $= \frac{1}{2}\ln\left(\frac{3}{1+2x}\right)$ A1

So $\dfrac{dy}{dx} = \frac{1}{2}\left(\frac{1+2x}{3}\right) \times \dfrac{-3\times 2}{(1+2x)^2}$ M1 $\Rightarrow \dfrac{dy}{dx} = \dfrac{-1}{(1+2x)}$ so $(2x+1)\dfrac{dy}{dx} + 1 = 0$ A1

Examiner comment

This is probably the simplest method, and follows from (a). Care needs to be taken over the algebra of simplifying the ln function, and the chain rule differentiation needs to be shown very clearly, since the answer is given.

Method 2

$\tanh(y) = \frac{1-x}{2+x}$ so differentiating gives

$\operatorname{sech}^2 y\,\dfrac{dy}{dx} = \dfrac{(2+x).(-1)-(1-x)}{(2+x)^2} = \dfrac{(-3)}{(2+x)^2}$ M1 A1 and $\operatorname{sech}^2 y = 1 - \tanh^2 y = 1 - \left(\dfrac{1-x}{2+x}\right)^2$

so $\left(1-\left(\dfrac{1-x}{2+x}\right)^2\right)\dfrac{dy}{dx} = \dfrac{(-3)}{(2+x)^2}$ M1 $\Rightarrow ((2+x)^2-(1-x)^2)\dfrac{dy}{dx} = -3$

$\Rightarrow (4+4x+x^2-1-x^2+2x)\dfrac{dy}{dx} = -3 \Rightarrow (3+6x)\dfrac{dy}{dx} + 3 = 0$

so $(2x+1)\dfrac{dy}{dx} + 1 = 0$ A1

Examiner comment

This method is a little more complicated, and dependent on candidates remembering the hyperbolic identity connecting tanh(y) and sech(y). This is partly balanced by a straightforward quotient rule differentiation, but as with method 1, care must be taken with the algebra, and all steps clearly shown as the answer is given. The first 2 marks are awarded for differentiating tanh(y) implicitly, and the second M1 for substituting for $\operatorname{sech}^2(y)$, and substituting for it. All working must be seen.

(c) Hence find the first three terms in the Maclaurin's series for $\tanh^{-1}\left(\frac{1-x}{2+x}\right)$ in the form $a \ln 3 + bx + cx^2$, where a, b and c are constants to be determined. [5]

When $x = 0$, $y = \frac{1}{2}\ln(3)$ using the result of (a) and $\frac{dy}{dx} = -1$ from (b) M1

Differentiating $(2x+1)\frac{dy}{dx} + 1 = 0$ leads to $2\frac{dy}{dx} + (2x+1)\frac{d^2y}{dx^2} = 0$ B1

Substituting for x and $\frac{dy}{dx}$ gives $2(-1) + \frac{d^2y}{dx^2} = 0$ so $\frac{d^2y}{dx^2} = 2$ M1

Putting these values into Maclaurin's series gives

$$y = \frac{1}{2}\ln(3) - x + x^2 + \ldots$$ M1 A1

Examiner comment

The differentiation can be done from the explicit expression for $\frac{dy}{dx}$ if Method 1 has been used, or implicitly if Method 2 has been followed. The B1 is for a correct second derivative. Detailed steps are essential as the structure of the answer has been given so M1 is awarded for the working associated with finding the values of y and the first and second derivative when $x = 0$, and the second M1 for substituting into Maclaurin's series.

Question 8

8 (a)(i) Find the set of values of a for which the system of equations

$$x - 2y - 2z + 7 = 0,$$
$$2x + (a-9)y - 10z + 11 = 0,$$
$$3x - 6y + 2az + 29 = 0,$$

has a unique solution. [4]

If there is a unique solution, $\left|\begin{pmatrix} 1 & -2 & -2 \\ 2 & a-9 & -10 \\ 3 & -6 & 2a \end{pmatrix}\right| \neq 0$

So $(1)(2a(a-9) - 60) - (-2)(2.2a - (-30)) + (-2)(2.(-6) - 3(a-9)) \neq 0$ M1 A1

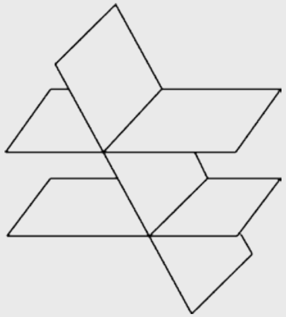
$2a^2 - 18a - 60 + 8a + 60 + 24 + 6a - 54 \neq 0$

$2a^2 - 4a - 30 \neq 0 \Rightarrow a^2 - 2a - 15 \neq 0$ M1 $\Rightarrow (a-5)(a+3) \neq 0$ so $a \neq 5, a \neq -3$ A1

Examiner comment

Candidates should show the working needed to find the determinant of the 3 × 3 matrix for this system of equations for the first two marks. They then need solve the resulting quadratic equation (M1) and to be careful to show the correct set of values which make the determinant non-zero (A1).

(a)(ii) Given that $a = -3$, show that the system of equations in part **(i)** has no solution. Interpret this situation geometrically. [3]



If $a = -3$, the third equation becomes $3x - 6y - 6z + 29 = 0$, but multiplying the first equation by 3 gives $3x - 6y - 6z + 21 = 0$, i.e. the equations are inconsistent. **B1** *The two planes represented by these equations are parallel, but not coincident.* **B1** *The third equation represents a non-parallel plane, so intersects each of the other two planes in a line.* **B1**

Examiner comment

Having checked the relationship between the three equations when a = -3, candidates need to state which pair of equations are inconsistent for the first mark, that these planes are parallel for the second mark. Recognising that the third equation represents a non-parallel plane which intersects each of the other two planes in a line gains the final mark. A diagram helps to clarify this final statement if candidates find it difficult to describe the situation in words.

(b) The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}.$$

(b)(i) Find the eigenvalues of **A**. [4]

The characteristic equation is $\left|\begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix}\right| = 0$ B1

$$(1-\lambda)((2-\lambda)(3-\lambda) - 1.2) - 1(0 - (-1)(2)) + 2(0 - (-1)(2-\lambda)) = 0$$

$$(1-\lambda)(2-\lambda)(3-\lambda) - 2(1-\lambda) - 2 + 4 - 2\lambda = 0$$

$$(1-\lambda)(2-\lambda)(3-\lambda) = 0 \ (*)$$ M1

$\lambda = 1, 2$ or 3 A1 A1

Examiner comment

Candidates need to show that the determinant of **A** $-\lambda$**I** = 0, then evaluate this determinant and simplify the resulting expression for the method mark. The three values will always be distinct (and non-zero) so some self-checking is possible. Two out of three solutions correct would earn A1.

(b)(ii) Use the characteristic equation of **A** to find $\mathbf{A}^{-1}$. [4]

Multiplying out the characteristic equation (*) from (i) gives

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Since **A** satisfies its characteristic equation, so $\mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} - 6\mathbf{I} = 0$ B1

Multiplying through by $\mathbf{A}^{-1}$ gives $\mathbf{A}^2 - 6\mathbf{A} + 11\mathbf{I} - 6\mathbf{A}^{-1} = 0$ M1

So $6\mathbf{A}^{-1} = \mathbf{A}^2 - 6\mathbf{A} + 11\mathbf{I}$ and $\mathbf{A}^2 = \begin{pmatrix} -1 & 5 & 10 \\ -2 & 6 & 10 \\ -4 & 4 & 9 \end{pmatrix}$ so $\mathbf{A}^{-1} = \frac{1}{6}\begin{pmatrix} 4 & -1 & -2 \\ -2 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix}$ M1 A1

Examiner comment

The method is specified in the question, so candidates must recall and use the fact that a matrix satisfies its characteristic equation (earning the B1) and manipulate this equation to find $\mathbf{A}^{-1}$ for the first method mark. Other methods will not be credited in this case. It will be expected to see the matrix for $\mathbf{A}^2$ as it is an essential component of the equation, so the final line will earn M1 and A1 if fully correct.

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