Cambridge Assessment International Education

Specimen Paper Answers

Paper 1

Cambridge International AS & A Level Further Mathematics 9231

For examination from 2020

Version 1

Cambridge Pathway

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Contents

Introduction

The main aim of this booklet is to exemplify standards for those teaching Cambridge International AS & A Level Further Mathematics 9231, and to show examples of model answers to the 2020 Specimen Paper 1. Paper 1 assesses the syllabus content for Further Pure Mathematics 1. We have provided answers for each question in the specimen paper, along with examiner comments explaining where and why marks were awarded. Candidates need to demonstrate the appropriate techniques, as well as applying their knowledge when solving problems.

You will need to use the mark scheme alongside this document. This can be found on the School Support Hub (www.cambridgeinternational.org/support) on the 'Syllabus materials' tab – scroll down to the bottom of the page where the Specimen Paper materials are.

Individual examination questions may involve ideas and methods from more than one section of the syllabus content for that component. The main focus of examination questions will be the AS & A Level Further Mathematics syllabus content. However, candidates may need to make use of prior knowledge and mathematical techniques from previous study, as listed in the introduction to section 3 of the syllabus.

There are six to eight structured questions in Paper 1; candidates must answer **all** questions. Questions are of varied lengths and often contain several parts, labelled (a), (b), (c), which may have sub-parts (i), (ii), (iii), as needed. Some questions might require candidates to sketch graphs or diagrams, or draw accurate graphs.

Candidates are expected to answer directly on the question paper. All working should be shown neatly and clearly in the spaces provided for each question. New questions often start on a fresh page, so more answer space may be provided than is needed. If additional space is required, candidates should use the lined page at the end of the question paper, where the question number or numbers must be clearly shown.

Past exam resources and other teacher support materials are available on the School Support Hub (www.cambridgeinternational.org/support).

Assessment overview

There are three routes for Cambridge International AS & A Level Further Mathematics. Candidates may combine components as shown below.

Route 1 **AS Level only** (Candidates take the AS components in the same series)	**Paper 1** **Further Pure Mathematics 1**	**Paper 2** **Further Pure Mathematics 2**	**Paper 3** **Further Mechanics**	**Paper 4** **Further Probability & Statistics**
Either	✓	Not available for AS Level	✓	
Or	✓			✓

Route 2 **A Level (staged over two years)**	**Paper 1** **Further Pure Mathematics 1**	**Paper 2** **Further Pure Mathematics 2**	**Paper 3** **Further Mechanics**	**Paper 4** **Further Probability & Statistics**
Either Year 1 AS Level	✓		✓	
Year 2 Complete the A Level		✓		✓
Or Year 1 AS Level	✓			✓
Year 2 Complete the A Level		✓	✓	

Route 3 **A Level** (Candidates take the A Level components in the same series)	**Paper 1** **Further Pure Mathematics 1**	**Paper 2** **Further Pure Mathematics 2**	**Paper 3** **Further Mechanics**	**Paper 4** **Further Probability & Statistics**
Year 2 full A Level	✓	✓	✓	✓

Paper 1 – Further Pure Mathematics 1

- Written examination, 2 hours, 75 marks
- 6 to 8 structured questions based on the Further Pure Mathematics 1 subject content
- Candidates answer all questions
- Externally assessed by Cambridge International
- 60% of the AS Level
- 30% of the A Level

This is compulsory for AS Level and A Level.

Assessment objectives

The assessment objectives (AOs) are the same for all papers:

AO1 Knowledge and understanding

- Show understanding of relevant mathematical concepts, terminology and notation
- Recall accurately and use appropriate mathematical manipulative techniques

AO2 Application and communication

- Recognise the appropriate mathematical procedure for a given situation
- Apply appropriate combinations of mathematical skills and techniques in solving problems
- Present relevant mathematical work, and communicate corresponding conclusions, in a clear and logical way

Weightings for assessment objectives

The approximate weightings (± 5%) allocated to each of the AOs are summarised below.

Assessment objectives as an approximate percentage of each component

Assessment objective	Weighting in components %			
	Paper 1	**Paper 2**	**Paper 3**	**Paper 4**
AO1 Knowledge and understanding	45	45	45	45
AO2 Application and communication	55	55	55	55

Assessment objectives as an approximate percentage of each qualification

Assessment objective	Weighting in AS Level %	Weighting in A Level %
AO1 Knowledge and understanding	45	45
AO2 Application and communication	55	55

Question 1

1 **(a)** Given that $f(r) = \frac{1}{(r+1)(r+2)}$, show that

$$f(r-1) - f(r) = \frac{2}{r(r+1)(r+2)}$$

[2]

$$f(r-1) - f(r) = \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} \quad \text{M1}$$

$$= \frac{r+2-r}{r(r+1)(r+2)} \qquad = \frac{2}{r(r+1)(r+2)} \quad \text{A1}$$

Examiner comment

As this is an answer given (AG) question, there must be sufficient working to show how the expression is reached.

The accuracy mark (A1) depends on seeing the correct step before the final answer, as well as the correct answer.

1 **(b)** Hence find $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$ [3]

$2 \sum_1^n \frac{1}{r(r+1)(r+2)}$

$= \sum_1^n f(r-1) - f(r) = \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{2.3} - \frac{1}{3.4} + \frac{1}{3.4} - \frac{1}{4.5} + \ldots$

$\ldots + \frac{1}{(n-1)n} - \frac{1}{n(n+1)} + \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}$ M1 A1

So $\sum_1^n \frac{1}{r(r+1)(r+2)} = \frac{1}{2}\left(\frac{1}{1.2} - \frac{1}{(n+1)(n+2)}\right) = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$ B1

Examiner comment

The method mark (M1) is for setting up the method of differences, and the accuracy mark (A1) is for showing sufficient terms to justify the cancellation as well as correct cancellation.

It is important to show enough terms to establish the pattern; terms at both the start and the end of the list are needed.

The final mark (B1) is for the expression, which can be given in a different format to that shown.

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$ [1]

As $n \to \infty$, $\frac{1}{2(n+1)(n+2)} \to 0$

so $\sum_1^n \frac{1}{r(r+1)(r+2)} \to \frac{1}{4}$

Examiner comment

Follow through is allowed on *their* answer to (b) as long as it is positive.

In this example there is only one mark awarded, B1, for a correct answer only. In other questions, two marks might be available, so it is good practice to encourage learners to use full explanations.

Question 2

The mark scheme gives two alternative solutions for this question, so we have provided a model answer for both.

Method 1

2 It is given that $\phi(n) = 5^n(4n+1) - 1$, for $n = 1, 2, 3, \ldots$.

Prove, by mathematical induction, that $\phi(n)$ is divisible by 8 for every positive integer n. [7]

Let P_n be the proposition that $\phi(n)$ is divisible by 8

Then $\phi(1) = 5^1(4 \times 1 + 1) - 1 = 25 - 1 = 24$ which is divisible by 8, so P_1 is true. B1

Assume that for some fixed integer value of n, say k, P_k is true, i.e. $\phi(k) = 8l$ B1

Then $\phi(k+1) - \phi(k) = 5^{k+1}(4(k+1)+1) - 1 - (5^k(4k+1) - 1)$ M1

$= 5^{k+1}(4k+5) - 5^k(4k+1) = 5^k(5 \times 4k + 25 - 4k - 1)$

$= 5^k(16k+24) = 8 \times 5^k(2k+3) = 8m$ A1

So $\phi(k+1) = 8m + \phi(k) = 8m + 8l = 8(m+l)$ A1 which is divisible by 8. A1

Hence, since P_1 is true, and P_k being true implies P_{k+1} is also true, by the principle of mathematical induction, P_n is true for all positive integers. A1

Examiner comment

It is important to state the stages of an induction proof very clearly, using well-defined propositions. The first case must be checked, this can be done before or after the inductive stage.

There are often several ways of proving the inductive step. Whichever method is chosen, it must be complete, with enough working and a logical conclusion that does not make any assumptions about, for example, divisibility.

A summarising statement should link the two stages clearly.

Method 2

2 It is given that $\phi(n) = 5^n(4n+1) - 1$, for $n = 1, 2, 3, \ldots$.

Prove, by mathematical induction, that $\phi(n)$ is divisible by 8 for every positive integer n. [7]

Let P_n be the proposition that $\phi(n)$ is divisible by 8

Then $\phi(1) = 5^1(4 \times 1 + 1) - 1 = 25 - 1 = 24$ which is divisible by 8, so P_1 is true B1

Assume that for some fixed integer value of n, say k, P_k is true, i.e. $\phi(k) = 8l$ B1

Then $\phi(k+1) = 5^{k+1}(4(k+1)+1) - 1 = 5^{k+1}(4k+5) - 1$

$= 5(4k \times 5^k) + 25 \times 5^k - 1$ A1 M1

A1:

$= 5(8l - 5^k + 1) + 25 \times 5^k - 1$

$= 40l + 4 \times 5 \times 5^k + 4$

$= 40l + 4(2m)$

since 5^{k+1} is odd, so $5^{k+1} + 1$ is even. Hence $\phi(k+1)$ is divisible by 8. A1

Hence, since P_1 is true, and P_k being true implies P_{k+1} is also true, by the principle of mathematical induction, P_n is true for all positive integers. A1

Examiner comment

It is important to state the stages of an induction proof very clearly, using well-defined propositions. The first case must be checked – this can be done before or after the inductive stage.

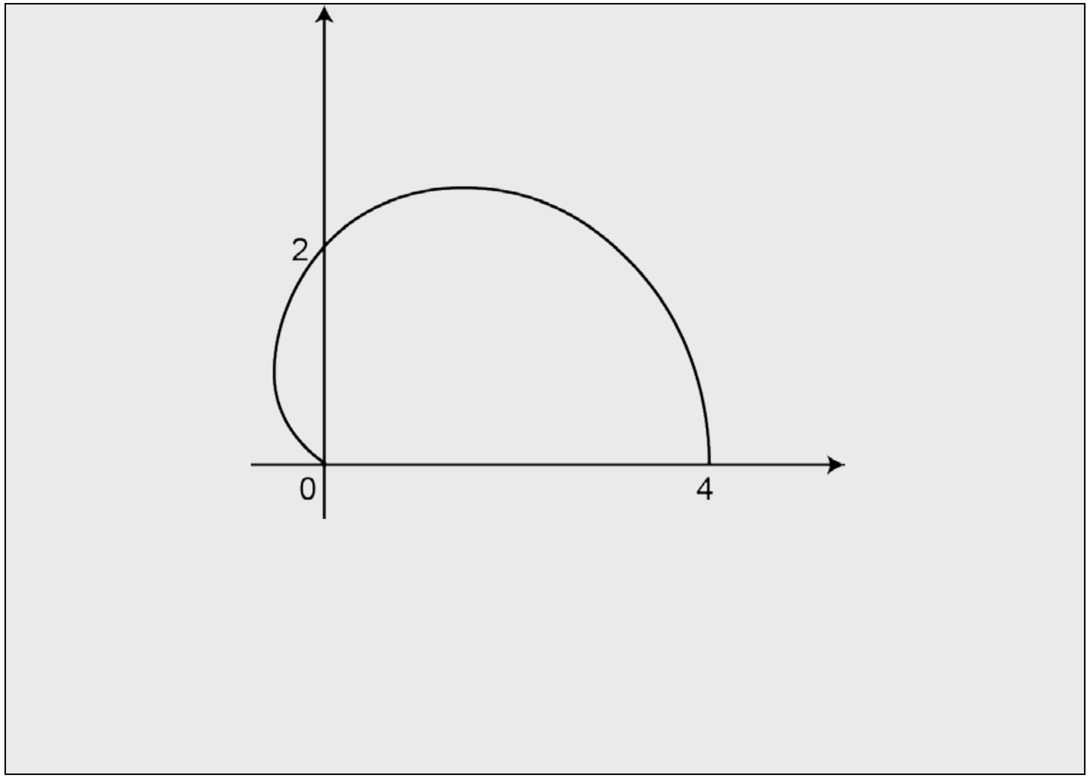
There are often several ways of proving the inductive step, whichever method is chosen it must be complete, with enough working and a logical conclusion that does not make any assumptions about, for example, divisibility.

A summarising statement should link the two stages clearly.

Question 3

3 The curve C has polar equation $r = 2 + 2\cos\theta$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C. [3]



Examiner comment

B1 for showing (4, 0) and (0, π) lie on C.

B1 for section $\frac{\pi}{2} < \theta < \pi$ correct

B1 Correct shape with curve perpendicular to initial line at (4,0)

It is important that the key points are clearly marked – this includes the intercepts with the initial line. It is often useful to include the intercept with $\theta = \dfrac{\pi}{2}$ as well.

The shape of the curve is important, particularly at the pole and when r is at a maximum.

In this example, $r \geqslant 0$ for all values of θ in the domain. In other cases, parts of the domain may give r <0, and these points must be excluded.

The Cartesian axes are **not** needed – an initial line is sufficient as long as all key points are labelled.

(b) Find the area of the region enclosed by C and the initial line. [4]

$$\text{Area} = \frac{1}{2}\int_0^{\pi}(2+2\cos\theta)^2\,d\theta \quad \text{M1}$$

$$= \int_0^{\pi}(2+4\cos\theta+2\cos^2\theta)\,d\theta = \int_0^{\pi}(3+4\cos\theta+\cos 2\theta)\,d\theta \quad \text{M1}$$

$$= \left[3\theta+4\sin\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi} = 3\pi \quad \text{A1 A1}$$

Examiner comment

The first M1 requires a completely correct integration, including limits, whilst the second M1 is for a good attempt to use the double angle formula to replace the $\cos^2\theta$ in the integrand. The first A1 is for integrating their expression (with three terms), and the final A1 for the correct answer.

(c) Show that the Cartesian equation of C can be expressed as $4(x^2+y^2)=(x^2+y^2-2x)^2$ [3]

$$r=\sqrt{x^2+y^2} \text{ and } x = r\cos\theta \Rightarrow \cos\theta = \frac{x}{\sqrt{x^2+y^2}} \quad \text{M1}$$

$$\text{so } \sqrt{x^2+y^2} = 2+2\frac{x}{\sqrt{x^2+y^2}} \quad \text{A1}$$

$$x^2+y^2 = 2\sqrt{x^2+y^2}+2x \Rightarrow 2\sqrt{x^2+y^2} = (x^2+y^2-2x) \Rightarrow \quad \text{A1}$$

$$4(x^2+y^2) = (x^2+y^2-2x)^2$$

Examiner comment

The method mark is awarded for using **both** relationships, as seen in the first line of working here, and a good attempt at substituting to find the Cartesian equation.

The first A1 is for the complete replacement of r and θ. The final A1 is awarded for rearranging and squaring both sides correctly. As the answer is given in the question, the penultimate line (or equivalent) of working is essential.

Question 4

The mark scheme gives two alternative solutions for this question, so we have provided a model answer for both.

4 The cubic equation

$$z^3 - z^2 - z - 5 = 0$$

has roots α, β and γ.

(a) Show that the value of $\alpha^3 + \beta^3 + \gamma^3$ is 19. [4]

Method 1

Rearranging equation gives $z^3 = z^2 + z + 5$ so $\sum\alpha^3 = \sum\alpha^2 + \sum\alpha + 3 \times 5$ M1

But $\sum\alpha = 1$ and $\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha = (1)^2 - 2(-1) = 3$ M1 A1

So $\sum\alpha^3 = 3 + 1 + 15 = 19$ A1

Method 2

$\sum\alpha^3 = (\sum\alpha)^3 - 3\sum\alpha\beta\sum\alpha + 3\alpha\beta\gamma$ M1

and $\sum\alpha = 1$, $\sum\alpha\beta = -1$ and $\alpha\beta\gamma = 5$ B1

So $\sum\alpha^3 = (1)^3 - 3 \times (-1) \times (1) + 3 \times 5 = 19$ M1 A1

Examiner comment

The first method mark is for rearranging the cubic, whilst the second method mark and the first A mark are for finding $\sum\alpha$ and $\sum\alpha^2$. The final A mark is dependent on seeing correct working, because this in as answer given (AG) question. This method only depends on recalling or recreating the formula for $\sum\alpha^2$, giving it an advantage over method 2 (shown below).

Examiner comment

The formula for $\sum\alpha^3$ is sometimes learnt by candidates, though recall is not always reliable. Most candidates are able to use the coefficients to find the three components needed. Since the answer is given, it is important to show all steps of working – the last A1 will not be awarded unless the substitution is shown.

(b) Find the value of $\alpha^4 + \beta^4 + \gamma^4$. [2]

Using the rearrangement above, and multiplying throughout by z gives $z^3 = z^2 + z + 5 \Rightarrow z^4 = z^3 + z^2 + 5z$ M1

so $\sum\alpha^4 = \sum\alpha^3 + \sum\alpha^2 + 5\sum\alpha = 19 + 3 + 5 = 27$ A1

Examiner comment

This solution follows very easily from the first method, as the rearrangement has already been done and the components calculated.

(c) Find a cubic equation with roots $\alpha + 1$, $\beta + 1$ and $\gamma + 1$, giving your answer in the form

$$px^3 + qx^2 + rx + s = 0,$$

where p, q, r and s are constants to be determined. [3]

$x = z + 1 \Rightarrow z = x - 1$ B1

and substituting gives $(x - 1)^3 - (x - 1)^2 - (x - 1) - 5 = 0$ M1

so $x^3 - 3x^2 + 3x - 1 - (x^2 - 2x + 1) - x + 1 - 5 = 0$

$\Rightarrow x^3 - 4x^2 + 4x - 6 = 0$ A1

Examiner comment

Substitutions will not be given in simple cases, so the first B1 is for recognising the appropriate substitution. M1 is awarded for making the substitution and A1 for simplifying the resulting cubic equation correctly.

Question 5

5 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 5 & k \\ -3 & -4 \end{pmatrix}.$$

(a) Find the value of k for which **A** is singular. [2]

> Det (A) = $5 \times (-4) - (-3)k = -20 + 3k$ M1
>
> so Det (A) = $0 \Rightarrow k = \frac{20}{3}$ A1

Examiner comment

The method mark is for knowing that Det(A) has to be zero for a singular matrix and knowing how to calculate the determinant. The A mark is awarded for a correct calculation.

It is now given that $k = 6$ so that $\mathbf{A} = \begin{pmatrix} 5 & 6 \\ -3 & -4 \end{pmatrix}$

(b) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **A**. [6]

> Let $y = mx$ be the equation of the invariant line, M1
> and the point (t, mt) transform to the point (T, mT)
>
> so $\begin{pmatrix} 5 & 6 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} 5t+6mt \\ -3t-4mt \end{pmatrix} = \begin{pmatrix} T \\ mT \end{pmatrix}$ M1
>
> therefore $5t + 6mt = T$ (i) and $-3t - 4mt = mT$ (ii)
> Dividing (ii) by (i) gives $m = \frac{-3-4m}{5+6m}$ M1 A1
>
> so $5m + 6m^2 = -3 - 4m \Rightarrow 6m^2 + 9m + 3 = 0 \Rightarrow 2m^2 + 3m + 1 = 0$
> $\Rightarrow (2m+1)(m+1) = 0$ and $m = -\frac{1}{2}$ or -1 A1
> therefore the invariant lines are $y = -x$ and $2y = -x$ A1

Examiner comment

The first M1 mark is for setting up the correct format for the invariant line, the second M1 is for the matrix representation. M1 and A1 is for obtaining a correct equation in ‘m’.

A1 for obtaining the correct values of ‘m’ and the final A1 is for stating the equations of the two invariant lines.

(c) The triangle *DEF* in the *x*-*y* plane is transformed by **A** onto triangle *PQR*.

(i) Given that the area of triangle *DEF* is 10 cm², find the area of triangle *PQR*. [2]

Det (A) = −20 + 18 = −2 M1 so Area = 2 × 10 = 20 cm² A1

Examiner comment

This question requires recall of the connection between the determinant of the matrix and the scale factor needed to find the area of a shape after a matrix transformation.

(ii) Find the matrix which transforms triangle *PQR* onto triangle *DEF*. [2]

$A^{-1} = \frac{1}{-2}\begin{pmatrix} 5 & 6 \\ -3 & -4 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -\frac{3}{2} & \frac{-5}{2} \end{pmatrix}$ M1 A1

Examiner comment

The method mark is for knowing that the inverse matrix is required and using the correct method for finding the inverse, i.e. dividing by determinant, switching elements and signs with at most one error. The accuracy mark is for a completely correct matrix (equivalents are accepted).

Question 6

6 The position vectors of the points A, B, C, D are

$$2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}, \quad -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}, \quad \mathbf{i} + 4\mathbf{j} + \mathbf{k}, \quad \mathbf{i} + 5\mathbf{j} + m\mathbf{k},$$

respectively, where m is an integer. It is given that the shortest distance between the line through A and B and the line through C and D is 3.

(a) Show that the only possible value of m is 2. [7]

$\overrightarrow{AB} = -4\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\overrightarrow{CD} = \mathbf{j} + (m-1)\mathbf{k}$ and $\overrightarrow{AC} = -\mathbf{i} + 4\mathbf{k}$

so $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 1 & -1 \\ 0 & 1 & m-1 \end{vmatrix} = m\mathbf{i} + 4(m-1)\mathbf{j} - 4\mathbf{k}$ is the common perpendicular M1 A1

Using the formula for shortest distance between lines

$$\frac{\left|\begin{pmatrix} -1 \\ 0 \\ 4k \end{pmatrix} \cdot \begin{pmatrix} m \\ 4(m-1) \\ -4 \end{pmatrix}\right|}{\sqrt{m^2 + 16(m-1)^2 + 16}} = 3 \quad \text{so} \quad -m - 16 = 3\sqrt{m^2 + 16(m-1)^2 + 16}$$ M1 A1

$$m^2 + 32m + 256 = 9(17m^2 - 32m + 32) \text{ so } 152m^2 - 320m + 32 = 0$$

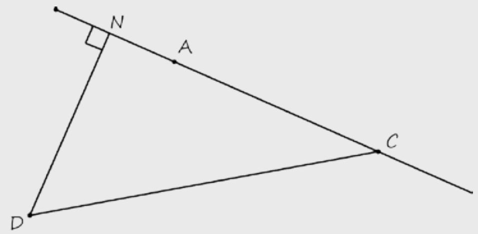
so $19m^2 - 40m + 4 = 0$ M1 A1 i.e. $(19m - 2)(m - 2) = 0$, as m is integer, $m = 2$ A1

Examiner comment

This method is usually well attempted by candidates. However, not all candidates remember the formula for shortest distance between lines correctly, and it is vital that the correct direction vectors are used.

6 (b) Find the shortest distance of D from the line through A and C.

[3]



If N is the foot of the perpendicular from D to AC then $|DN| = |CD|\sin C$

(or $|AD|\sin A$) but $|\overrightarrow{CA} \times \overrightarrow{CD}| = \left||\overrightarrow{CA}| \times |\overrightarrow{CD}|\sin C\right|$ so $|CD|\sin C = \left|\dfrac{|\overrightarrow{CA} \times \overrightarrow{CD}|}{|CA|}\right|$

$\overrightarrow{CA} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ B1 (Note that they could use AD instead)

$\overrightarrow{CA} \times \overrightarrow{CD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -4 \\ 0 & 1 & 1 \end{vmatrix} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$ M1

So $|DN| = |CD|\sin C = \dfrac{\sqrt{4^2+(-1)^2+1^2}}{\sqrt{1^2+0+(-4)^2}} = \dfrac{\sqrt{18}}{\sqrt{17}}$ A1

Examiner comment

There are many different ways to solve this part of the question. For example, some candidates find the value of the angle at *C* using the scalar product. For this method, the first mark (B1) is for identifying and finding the relevant vectors, the method mark is for using the cross product of *CA* and *CD*, and the final accuracy mark (A1) is for the correct answer. The mark scheme can be adapted to other methods.

Other canddiates find the coordinates of *N* using the equation of line *AC* and calculating the value of the parameter when *DN* is perpendicular to *AC* (using the scalar product).

6 (c) Show that the acute angle between the planes *ACD* and *BCD* is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$. [4]

The perpendicular ($\mathbf{n}_1$) to plane ACD is $\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$ from above.

The vector $\overrightarrow{CD}$ lies in the plane BCD as does the vector $\overrightarrow{BC} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$ B1

(note the candidate could use $\overrightarrow{BD}$)

so the perpendicular to plane BCD, $\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 5 \\ 0 & 1 & 1 \end{vmatrix} = \begin{pmatrix} -6 \\ -3 \\ 3 \end{pmatrix}$ M1

Using the scalar product: $\cos\theta = \frac{(4\mathbf{i} - \mathbf{j} + \mathbf{k}).(2\mathbf{i} + \mathbf{j} - \mathbf{k})}{\sqrt{16+1+1}\ \sqrt{4+1+1}}$ M1 $= \frac{1}{\sqrt{3}}$ so $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ A1

Examiner comment

Candidates might not realise that they have already found the vector perpendicular to *ACD*. Finding the perpendicular to plane *ABC* is straightforward, and most candidates will recall that the angle between the normals to the planes is equal to the angle between the planes, though a few might then think they have to subtract this from π or $\frac{\pi}{2}$.

Question 7

7 The curve C has equation $y = \dfrac{2x^2 - 3x - 2}{x^2 - 2x + 1}$

(a) State the equations of the asymptotes of C. [2]

$x^2 - 2x + 1 = (x-1)^2$ so $x = 1$ is vertical asymptote **B1**

and $y = 2$ is horizontal asymptote by inspection (or long division). **B1**

Examiner comment

The question says state the asymptotes, so working is not essential in this case, but most candidates will want to show their factorisation of the denominator to find the vertical asymptote. The horizontal asymptote can be seen because the order of numerator and denominator are the same, but again, some candidates may show a long division, or divide throughout by x^2.

(b) Show that $y \leq \dfrac{25}{12}$ at all points on C. [4]

Multiplying across the equation of C by the denominator, gives

$(x^2 - 2x + 1)y = 2x^2 - 3x - 2$ **M1**

Rearranging to form a quadratic in x^2 : $(y-2)x^2 - (2y-3)x + (y+2) = 0$ **A1**

For real x, $(2y-3)^2 - 4(y-2)(y+2) \geq 0 \rightarrow 4y^2 - 12y + 9 - 4y^2 + 16 \geq 0$ **M1**

$25 - 12y \geq 0$

so $y \leq \dfrac{25}{12}$ **A1**

Examiner comment

This is the most straightforward method for finding the range of y – finding stationary points can be used, but very often candidates do not present a complete argument to support their findings. Using this method leads to a fully reasoned solution.

Because the answer is given, candidates must give full working, so they need to show how they find the quadratic in x, then show that the discriminant must be ≥ 0 if x is real, hence arriving at the correct inequality for y.

(c) Find the coordinates of any stationary points of *C*. [3]

$$\frac{dy}{dx} = \frac{(x^2 - 2x + 1)(4x - 3) - (2x^2 - 3x - 2)(2x - 2)}{(x^2 - 2x + 1)^2}$$

so $\frac{dy}{dx} = 0$ when $(x^2 - 2x + 1)(4x - 3) - (2x^2 - 3x - 2)(2x - 2) = 0$ M1

$4x^3 - 11x^2 + 10x - 3 - (4x^3 - 10x^2 + 2x + 4) = 0 \Rightarrow -x^2 + 8x - 7 = 0 \Rightarrow (x-7)(x-1) = 0$

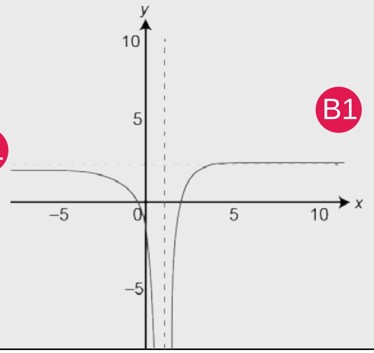
but $x = 1$ is an asymptote, so $x = 7$ is the only possible value. M1

Coordinates $(7, \frac{25}{12})$ A1

Examiner comment

It is also possible to differentiate after dividing the algebraic fraction. Candidates are usually secure on using the quotient rule. In this case, one solution is not valid as it is one of the asymptotes, so this solution must be rejected. Both coordinates are required for the final answer.

(d) Sketch C, stating the coordinates of any intersections of C with the coordinate axes and the asymptotes. [4]



B1 B1

Coordinates of intersections: $(0, -2)$, $(-0.5, 0)$, $(2, 0)$ and $(4, 2)$ B1

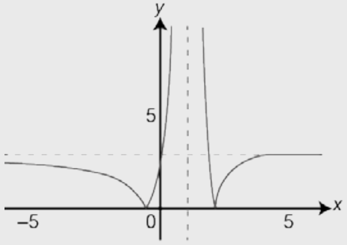
Asymptotes: $x = 1$ and $y = \frac{25}{12}$ B1

Examiner comment

All four intersections need to be marked, and the asymptotes either labelled on the graph or alongside.

(e) Sketch the curve with equation $y = \left|\frac{2x^2 - 3x - 2}{x^2 - 2x + 1}\right|$ and find the set of values of x for which $\left|\frac{2x^2 - 3x - 2}{x^2 - 2x + 1}\right| < 2$. [4]

B1



$\frac{2x^2 - 3x - 2}{x^2 - 2x + 1} < 2 \Rightarrow$

$2x^2 - 3x - 2 <$

$2x^2 - 4x + 2 \Rightarrow$

$x < 4$

$\frac{2x^2 - 3x - 2}{x^2 - 2x + 1} > -2 \Rightarrow$

$2x^2 - 3x - 2 < -2x^2 + 4x - 2$

$\Rightarrow 4x^2 < 7x \Rightarrow x(4x - 7) < 0 \Rightarrow x < 0$ B1 or $x > \frac{7}{4}$

So $x < 0$ or $\frac{7}{4} < x < 4$ M1 A1

Examiner comment

The graph follows on from part (d), so B1 is awarded if any parts of their graph below the x-axis are reflected in the x-axis.

Points of intersection with $y = 2$ and -2 need to be calculated and the correct sections identified.

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