

Scheme of Work

Cambridge International AS & A Level Further Mathematics 9231

Further Pure Mathematics 2 (for Paper 2)

For examination from 2020



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Introduction

This scheme of work has been designed to support you in your teaching and lesson planning. The scheme of work has been separated into four documents, one for each examination Paper: Further Pure Mathematics 1 (for Paper 1); Further Pure Mathematics 2 (for Paper 2); Further Mechanics (for Paper 3); and Further Probability & Statistics (for Paper 4). This document relates only to **Further Pure Mathematics 2 (for Paper 2)**.

Making full use of this scheme of work will help you to improve both your teaching and your learners' potential. It is important to have a scheme of work in place in order for you to guarantee that the syllabus is covered fully. You can choose what approach to take and you know the nature of your institution and the levels of ability of your learners. What follows is just one possible approach you could take and you should always check the syllabus for the content of your course.

There is a separate table for each topic of the Further Pure Mathematics 2 syllabus content (2.1 Hyperbolic functions, 2.2 Matrices, etc.). Each of the bullet points from the syllabus subject content are listed along with teaching suggestions. There is a 'Main theme' for each, which is the focus activity/activities for the content. Where possible, this is supported by an 'Introduction' activity to set the context. Suggestions for independent study (**I**) and formative assessment (**F**) are also included. Opportunities for differentiation are indicated as **Extension activities**; there is the potential for differentiation by resource, grouping, expected level of outcome, and degree of support by teacher, throughout the scheme of work. Timings for activities and feedback are left to the judgment of the teacher, according to the level of the learners and size of the class. Length of time allocated to a task is another possible area for differentiation.

Key concepts

This scheme of work is underpinned by the assumption that Mathematics is fundamentally problem solving and representing systems and models in different ways. The key concepts are highlighted as a separate item in the new syllabus and teachers should be aware that learners will be assessed on their direct knowledge and understanding of the same. Learners should be able to describe and explain the key concepts as well as demonstrate their ability to apply them to novel situations and evaluate them. They are not referred to specifically in the Scheme of Work as they are essential to tackling problems in all topics.

The key concepts are as follows:

- **Problem solving**

Mathematics is fundamentally problem solving and representing systems and models in different ways.

These include:

- Algebra: this is an essential tool which supports and expresses mathematical reasoning and provides a means to generalise across a number of contexts.
- Geometrical techniques: algebraic representations also describe a spatial relationship, which gives us a new way to understand a situation.
- Calculus: this is a fundamental element which describes change in dynamic situations and underlines the links between functions and graphs.
- Mechanical models: these explain and predict how particles and objects move or remain stable under the influence of forces.
- Statistical methods: these are used to quantify and model aspects of the world around us. Probability theory predicts how chance events might proceed, and whether assumptions about chance are justified by evidence.

- **Communication**

Mathematical proof and reasoning is expressed using algebra and notation so that others can follow each line of reasoning and confirm its completeness and accuracy. Mathematical notation is universal. Each solution is structured, but proof and problem solving also invite creative and original thinking.

- **Mathematical modelling**

Mathematical modelling can be applied to many different situations and problems, leading to predictions and solutions. A variety of mathematical content areas and techniques may be required to create the model. Once the model has been created and applied, the results can be interpreted to give predictions and information about the real world.

Recommended prior knowledge

Knowledge of the Cambridge IGCSE® Mathematics 0580 syllabus (or equivalent) is required for the Cambridge International AS & A Level Further Mathematics 9231 course. All topics from the Cambridge International AS & A Level Mathematics 9709 course are also needed as prior knowledge. However, it is possible to teach both A Level courses alongside each other in parallel, as not all of the Further Mathematics topics have direct dependencies. See *Parallel teaching – A two-year plan to co-teach Cambridge International AS & A Level Mathematics 9709 and Cambridge International AS & A Level Further Mathematics 9231* for guidance. This is available on the School Support Hub www.cambridgeinternational.org/support

Guided learning hours

Guided learning hours give an indication of the amount of contact time teachers need to have with learners to deliver a particular course. Our syllabuses are designed around 180 hours for Cambridge International AS Level, and 360 hours for Cambridge International A Level. The number of hours will vary depending on local practice and your learners' previous experience of the subject.

It is recommended that you spend about 90 hours teaching the content for each Paper: Further Pure Mathematics 1 (for Paper 1); Further Pure Mathematics 2 (for Paper 2); Further Mechanics (for Paper 3); and Further Probability & Statistics (for Paper 4).

The table below gives some guidance about how many hours are recommended for each topic within Further Pure Mathematics 2 (for Paper 2).

Suggested teaching order of syllabus content	Hours
2.1 Hyperbolic functions	10
2.3 Differentiation	15
2.4 Integration	15
2.5 Complex numbers	15
2.2 Matrices	15
2.6 Differential equations	20

Resources

The textbooks endorsed by Cambridge International for use with this course are listed at www.cambridgeinternational.org. Endorsed textbooks have been written to be closely aligned to the syllabus they support, and have been through a detailed quality assurance process. As such, all textbooks endorsed by Cambridge International for this syllabus are the ideal resource to be used alongside this scheme of work as they cover each learning objective. There is also a support resource available for co-teaching the Cambridge International AS & A Level Further Mathematics 9231 course alongside the Cambridge International AS & A Level Mathematics 9709 course: *Parallel teaching – A two-year plan to co-teach Cambridge International AS & A Level Mathematics 9709 and Cambridge International AS & A Level Further Mathematics 9231*, which is available on the School Support Hub.

School Support Hub

The School Support Hub (www.cambridgeinternational.org/support) is a secure online resource bank and community forum for Cambridge teachers, where you can download specimen and past question papers, mark schemes and other resources. We also offer online and face-to-face training; details of forthcoming training opportunities are posted online. This scheme of work is available as PDF and an editable version in Microsoft Word format from the School Support Hub. If you are unable to use Microsoft Word you can download Open Office free of charge from www.openoffice.org.

Websites

This scheme of work includes website links providing access to internet resources. Cambridge Assessment International Education is not responsible for the accuracy or content of information contained in these sites. The inclusion of a link to an external website should not be understood to be an endorsement of that website or the site's owners (or their products/services).

The website pages referenced in this scheme of work were selected when the scheme of work was produced. Other aspects of the sites were not checked and only particular resources are recommended.

Useful websites include:

www.stem.org.uk The National STEM Learning Network provides access to a range of resources.

<http://integralmaths.org> The Integral® website provides resources developed by a curriculum development project called MEI. Since these schemes were first written, this website has become available only through paid subscription.

www.mmlsoft.com/index.php/products/tarsia The TARSIA software is free to download. It can be used to download and create puzzles to test manipulation.

Important notice about past papers

The 2020 syllabus (for examination in 2020) includes changes to the assessment structure, assessment objective weightings and syllabus content when compared to the 2017–2018 and 2019 syllabuses. Therefore, if you use past papers and mark schemes from earlier series, please do so with caution. It is still possible to help your learners understand what the examination papers look like and to give an idea of the required standard but please be aware that some of the content, the assessment structure and nature of the mark scheme has changed. Please also use the specimen papers and mark schemes for the 2020 series.

How to get the most out of this scheme of work – integrating syllabus content, skills and teaching strategies

We have written this scheme of work for the Cambridge International AS & A Level Further Mathematics 9231 syllabus and it provides some ideas and suggestions of how to cover the content of the syllabus. We have designed the following features to help guide you through your course.

Syllabus subject content lists the subject content bullet points from the syllabus, making it clear the knowledge your learners need to build. Pass these on to your learners by expressing them as 'We are learning to / about...'.
(I)

Suggested teaching activities give you lots of ideas about how you can present learners with new information. Try more active methods that get your learners motivated and practising new skills. Where possible, the activities are separated into 'Introduction' ideas to set the context, and 'Main themes' that form the core of the teaching.
(I)

Syllabus subject content

use the method of mathematical induction to establish a given result;

Suggested teaching activities

Introduction: To put this topic in context, give some examples of deductive proofs, such as proving the formula for the sum of an algebraic or geometric series; or the quadratic formula. It is important to make sure learners understand the need to show mathematical rigour at every step. (I)

Extension activity: Proof by contradiction and by exhaustion could also be examined as an extension activity from the same resource. Some classical proofs are illustrated on <https://undergroundmathematics.org>, search for 'Divisibility & Induction'.
(I)

Main theme: Proving familiar results such as the sum of the positive integers is a good place to start. Some useful examples and exercises can be found on the Integral website (<http://integralmaths.org>). There is a good matching activity on the STEM website (www.stem.org.uk) called 'Creating Connections Between Topics: Proof by Induction'. (F)

Divisibility tests, inequalities, calculus, geometry and series may all be used as contexts and it's always important that the deductive step is written out fully with a rigorous argument. Note: some questions involving calculus may require techniques from Cambridge International AS & A Level Mathematics (9709) Pure Mathematics 3, so take this into consideration when planning.
(F)

Extension activities provide your more able learners with further challenge beyond the basic content of the course. Innovation and independent learning are the basis of these activities.
(I)

Independent study (I) gives your learners the opportunity to develop their own ideas and understanding with direct input from you.
(I)

Past and specimen papers

Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support (F)
Specimen Paper 1 Q2 Divisibility
Nov 2016 Paper 11 Q4 Properties of factorials

Past papers, specimen papers and mark schemes are available for you to download at:
www.cambridgeinternational.org/support

Using these resources with your learners allows you to check their progress and give them confidence and understanding.

Formative assessment (F) is on-going assessment that informs you about the progress of your learners. Don't forget to leave time to review what your learners have learnt, you could try question and answer, tests, quizzes, 'mind maps', or 'concept maps'.
(F)

2.1 Hyperbolic functions

Syllabus subject content	Suggested teaching activities
<p>Understand the definitions of the hyperbolic functions $\sinh x$, $\cosh x$, $\tanh x$, $\operatorname{sech} x$, $\operatorname{cosech} x$ and $\coth x$ in terms of the exponential function.</p> <p>Sketch the graphs of hyperbolic functions.</p>	<p>Prior knowledge: Learners need to have worked with the exponential function and be confident in handling exponentials from their work on Pure Mathematics 3 from the Cambridge International AS & A Level Mathematics 9709. They should also know the definitions of the cosec, sec and cotan functions from Pure Mathematics 1.</p> <p>Main theme: The definitions for $\sinh x$ and $\cosh x$ need to be given and learned. Learners can use these definitions to write down the other expressions from their knowledge of trigonometry. Brief notes on these functions can be found in the <i>Further Pure Mathematics 3 Revision notes</i> (2016) in the A level student area of the Mr Barton's Maths website (www.mrbartonmaths.com) – enter the student site, then 'A Level', then 'Notes, Videos and Examples' – the revision notes are listed under the heading 'Further Pure 3'. Videos can be found on these expressions, as well as the concepts that follow, on the Exam Solutions website (www.examsolutions.net). For example, click on 'A Level' then 'OCR', and the link for 'FP2 Tutorials'.</p> <p>Learners can sketch these functions from their knowledge of the definitions and graphs of e^x and e^{-x}. They can check their sketches with a free graphing package such as Geogebra (www.geogebra.org) or Desmos (www.desmos.com/), or using a graphic calculator. (I) This could be done as a flipped learning activity, with learners exploring the graphs using software and/or Youtube videos for homework, and then completing an exercise in class. (F)</p> <p>Two card-matching exercises can be found on the integral website (http://integralmaths.org) in the section on hyperbolic functions; for example, under MEI Further Pure Mathematics 2. A similar matching exercise is available on the TES website (www.tes.com): search for 'Hyperbolic Functions' and use the resource created by SR Whitehouse. (F)</p> <p>Extension activity: The activity 'From parabolas to catenaries' on Underground Mathematics (www.undergroundmathematics.org) looks at the equation of a catenary and combines hyperbolic functions with limiting processes.</p>

Syllabus subject content	Suggested teaching activities
<p>Prove and use identities involving hyperbolic functions; e.g.</p> $\cosh^2 x - \sinh^2 x \equiv 1$ $\sinh 2x \equiv 2\sinh x \cosh x,$ <p>and similar results corresponding to the standard trigonometric identities.</p>	<p>Main theme: The identities can be proved by learners using the definitions of the hyperbolic functions. The common identities are listed on the STEM website (www.stem.org.uk): search for 'Engineering Maths First Aid Kit', then in here, find the link for 'Functions and Graphs'. There are also notes and exercises, where learners have to prove identities, on the Centre for Innovation in Mathematics Teaching (CIMT) website (www.cimt.org.uk): click on 'Resources', then follow the link to the Mathematics Enhancement Programme (MEP) and then click on 'A-Level Course Material'. (F)</p> <p>Extension activity: Learners could find the rule that links hyperbolic identities with their corresponding trigonometric identities (Osborn's rule). This has a good link to complex numbers.</p>
<p>Understand and use the definitions of the inverse hyperbolic functions and derive and use the logarithmic forms</p>	<p>Main theme: Ask learners to work out the logarithmic forms of the inverse hyperbolic functions by rearranging the equations in exponential form to form quadratics in e^x and solving these. There is a Tarsia puzzle on the TES website (www.tes.com) that practices using this form: search for 'Puzzle using simple Hyperbolic Equations' and use the resource created by SR Whitehouse. (F)</p> <p>(I) For comprehensive notes on the whole topic, as well as exercises (F), see the CIMT website (www.cimt.org.uk): click on 'Resources', then follow the link to the 'Mathematics Enhancement Programme (MEP)', then 'A-Level Course Material' and then 'Hyperbolic functions'. Videos on this part of the topic, as well as earlier sections, can also be found on the Exam Solutions website (www.examsolutions.net). For example, click on 'A Level' then 'OCR', and the links for 'FP2 ...'.</p> <p>There are also hyperbolic dominos on the Integral website (http://integralmaths.org). Search under MEI Further Pure Mathematics 2' for 'Practising inverses and logarithmic forms'.</p>
Past and specimen examination papers	
<p>Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support (F)</p> <p>The 2020 syllabus includes changes to the assessment structure, assessment objective weightings and syllabus content when compared to the 2017–2018 and 2019 syllabuses. Therefore, if you use past papers and mark schemes from earlier series, please do so with caution. It is still possible to help your learners understand what the examination papers look like and to give an idea of the required standard but please be aware that some of the content, the assessment structure and nature of the mark scheme has changed. Please also use the specimen papers and mark schemes for the 2020 series.</p> <p>Specimen Paper 2 Q7(a) inverse hyperbolic</p>	

2.3 Differentiation

Syllabus subject content	Suggested teaching activities
Differentiate hyperbolic functions and differentiate $\sin^{-1}x$, $\cos^{-1}x$, $\sinh^{-1}x$, $\cosh^{-1}x$ and $\tanh^{-1}x$	<p>Prior knowledge: Learners should have previously differentiated $\tan^{-1}x$ and exponential functions.</p> <p>Main theme: Helping Engineers Learn Mathematics (HELM) is a free online series of 'workbooks' that is accessible from a number of university websites. For example, if you do a Google search for 'HELM Workbooks' you will find a link to pdf versions of each workbook on the University of Southampton website. There are detailed notes on this topic in <i>Workbook 11: Differentiation</i>.</p> <p>(F) There are a number of resources on differentiating hyperbolic functions, including dominoes to match functions with their derivatives, on the TES website (www.tes.com); use those created by SR Whitehouse. Two sets of dominoes are also available on the Integral website (http://integralmaths.org); search under MEI Further Pure Mathematics 2 'Practising integration and differentiation of hyperbolic functions'. You may need to install the TARSIA software described in the introduction to download some activities. Some good exercises can also be found on the 17calculus website (17calculus.com): click on 'Derivatives' and then 'Hyperbolic Functions'.</p> <p>Two sets of dominoes are available on the Integral website (www.integralmaths.com) under MEI FP2 practising integration and differentiation of hyperbolic functions.</p> <p>Extension: The article <i>Infinite Continued Fractions</i> discusses the work of Ramanujan (search Google for 'Infinite Continued Fractions NRICH' to find the article). The investigative approach is encouraged in the activities 'Gosh Cosh' and 'Hyperbolic thinking' (both Stage 5 activities) on the NRICH website (nrich.maths.org).</p> <p>Linking this topic to the first sub-topic in topic '2.4 Integration' reinforces the connection between functions and their derivatives. The two topics could be run in parallel or with '2.4 Integration' immediately following this section of work.</p>

Syllabus subject content	Suggested teaching activities
<p>Obtain an expression for $\frac{d^2y}{dx^2}$ in cases where the relation between x and y is defined implicitly or parametrically</p>	<p>Prior knowledge: Learners need to have the full range of calculus skills up to Pure Mathematics 3 level, including parametric and implicit differentiation.</p> <p>Main theme: This is a relatively straightforward application of earlier calculus skills, which requires a lot of stamina and precision. Learners need to practice these skills extensively to make sure they can handle the notation correctly and avoid common errors, particularly with the second derivatives.</p> <p>Some notes on second derivatives of parametric equations can be found on the Mathscentre website (www.mathcentre.ac.uk). Examples on implicit functions can be found on the Interactive Mathematics Site (www.intmath.com): click on 'Differentiation' then '9 Higher Derivatives'. Some suitable exercises can be found on 17calculus website (17calculus.com). (F)</p>
<p>Derive and use the first few terms of Maclaurin's series for a function.</p> <p>Note: Derivation of a general term is not included, but successive 'implicit' differentiation steps may be required, e.g. for $y = \tan x$ following an initial differentiation rearranged as $y' = 1 + y^2$</p>	<p>Prior knowledge: Learners need to be able to differentiate the full range of functions seen in Cambridge International AS & A Level Mathematics 9709 and Cambridge International AS & A Level Further Mathematics 9231, as well as implicitly.</p> <p>Main theme: Learners can explore the series expansion for e^x using the fact that $\frac{d}{dx}e^x = e^x$ and setting $x = 0$ after each stage of differentiation. This exploratory approach can be used for other simple series expansions too before arriving at the formula.</p> <p>There are notes and exercises (F) on the CIMT website (www.cimt.org.uk): click on 'Resources' then follow the link to the Mathematics Enhancement Programme (MEP), then 'A-Level Course Material', 'Further Pure Mathematics' and select '6. Sequences and Series'. There are notes on how to use series expansions to find limits on the Integral website (http://integralmaths.org) under 'AQA FP3', and how to use expansions to find approximate values under 'MEI FP2'.</p> <p>The Exam Solutions website (http://www.examsolutions.net) has explanatory videos on various expansions, and a set of worked examination questions. Search in the Maclaurin's Series section, under MEI FP2.</p>

Past and specimen examination papers

Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support (F)

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Specimen Paper 2 Q7(b) differentiating inverse hyperbolic

2.4 Integration

Syllabus subject content	Suggested teaching activities
<p>Integrate hyperbolic functions and recognise integrals of functions of the form</p> $\frac{1}{\sqrt{a^2 - x^2}}, \frac{1}{\sqrt{x^2 + a^2}}, \frac{1}{\sqrt{x^2 - a^2}},$ <p>and integrate associated functions using trigonometric or hyperbolic substitutions as appropriate</p>	<p>Prior knowledge: Learners should have covered the section on differentiation in Pure Mathematics 3 of the Cambridge International AS & A Level Mathematics 9709 course. They should also have integrated expressions of the form $\frac{1}{(1+x^2)}$, and be able to integrate by parts and substitution.</p> <p>Main theme: This work follows directly from the work in '2.4 Differentiation'. The standard results are in the Notation List; give learners the opportunity to use a copy of the Notation List as they work through examples so that they know where to find the results during the examinations. The Notation List for 2020 can be found on the School Support Hub (www.cambridgeinternational.org/support) on the page for Cambridge International AS & A Level Further Mathematics 9231 under 'Other documents'.</p> <p>Notes on integrating these particular forms of integrand can be found on the PPLATO: Promoting Physics Learning and Teaching Opportunities website (www.met.reading.ac.uk/pplato2). Scroll down to 'Mathematics' and then to 'MATH 5.5: Further Integration'. Learners might find the videos from the Further Mathematics Support Programme (FMSP) http://furthermaths.org.uk helpful. From the home page, click on 'Students', 'Revision', 'OCR Revision', 'Further Pure 2' and then 'Integration'. (I) Some useful exercises (F) can be found on the 17calculus website (17calculus.com) Click on 'Integrals' and then 'Trig Integration Practice'. It is important that learners complete the square if necessary so that they can compare their denominator.</p>
<p>Derive and use reduction formulae for the evaluation of definite integrals, e.g.</p> $\int_0^{\frac{1}{2}\pi} \sin^n x \, dx, \int_0^1 e^{-x}(1-x)^n \, dx$ <p>In harder cases hints may be given, e.g. $\int_0^{\frac{1}{4}\pi} \sec^n x \, dx$ by considering</p> $\frac{d}{dx}(\tan x \sec^n x)$	<p>Prior knowledge: A quick revision of integration by parts from Pure Mathematics 3 will set learners up for this topic.</p> <p>Main activity: A simple starting point such as $I = \int e^x \sin(x) \, dx$ where I re-appears on the right-hand side gives the idea for a reduction formula before embarking on a more general case. Reduction formulae often require the integrand to be split to create a multiple of the original I. These splits may be algebraic (splitting $f(x)^{\frac{3}{2}}$ into $f(x) \times f(x)^{\frac{1}{2}}$) or trigonometric (for example re-writing $\cos^2 x$ as $1 - \sin^2 x$). Often, when a question starts with a differentiation, using the derivative will inform the choice of integrand. There are two collections of past exam questions (F) on the A Level section of Mr Barton's Maths website (www.mrbartonmaths.com) under Further Pure 2.</p>
<p>Understand how the area under a curve may be approximated by</p>	<p>Main theme: The Integration video on the FMSP website (http://furthermaths.org.uk) goes through an example of deriving an inequality. (From the home page, click on 'Students', 'Revision', 'OCR Revision', 'Further Pure 2' and</p>

Syllabus subject content	Suggested teaching activities
<p>areas of rectangles, and use rectangles to estimate or set bounds for the area under a curve or to derive inequalities or limits concerning sums</p> <p>Questions may involve either rectangles of unit width or rectangles whose widths can tend to zero, e.g.</p> $1 + \ln n > \sum_{r=1}^n \frac{1}{r} > \ln(n+1),$ $\sum_{r=1}^n \frac{1}{n} \left(1 + \frac{r}{n}\right)^{-1} \approx \int_0^1 (1+x)^{-1} dx$	<p>then ‘Integration’.) The Integral website (http://integralmaths.org) has some useful worked examples under ‘Numerical Methods’ in the section on OCR Further Pure 2.</p> <p>A useful set of OCR examination questions on this topic can be found on the Physics and Maths Tutor website (www.pmt.physicsandmathstutor.com). Click on ‘Maths Papers’, then under the heading ‘FP2’ click on ‘Questions by topic’, then under the heading ‘For OCR’ click on ‘FP2 Areas using Rectangles’. (F)</p> <p>Extension activity: Learners might be interested to read the article <i>An infinite series of surprises</i> from the Plus magazine website (https://plus.maths.org/content). This is good for historical context.</p>
<p>Use integration to find</p> <ul style="list-style-type: none"> – arc lengths for curves with equations in Cartesian coordinates, including the use of a parameter, or in polar coordinates – surface areas of revolution about one of the axes for curves with equations in Cartesian coordinates, including the use of a parameter. <p>Any questions involving integration may require techniques from Cambridge International A Level Mathematics (9709) applied to more difficult cases, e.g. integration by parts for $\int e^x \sin x \, dx$, or use of the substitution $t = \tan \frac{1}{2} x$.</p> <p>Surface areas of revolution for curves with equations in polar coordinates will not be required.</p>	<p>Prior knowledge: To be able to answer questions on this topic, learners will need to have knowledge of the whole range of calculus techniques, including reduction formulae.</p> <p>Main theme: Learners need to memorise all the formulae, so a lot of learning is required. It’s a good idea to go through step-by-step proofs for each formula, as this gives learners a better chance to work out the formulae if their memory lets them down in an examination.</p> <p>There are notes on these topics in the Community Resources on the School Support Hub (www.cambridgeinternational.org/support) as they are not often covered in textbooks. Notes and examples of some applications can also be found in the Just the Maths booklets. ‘Just the Maths’ is a collection of units written by A J Hobson that is available online (https://archive.uea.ac.uk/jtm).</p> <p>Extension activity: Sixth Term Examination Paper (STEP) is a well-established mathematics examination. There are a number of STEP questions that involve advanced integration techniques and provide useful challenges to learners. They can be found in various locations, including the Cambridge Assessment Admissions Testing website (www.admissionstesting.org). Click on ‘For test-takers’, then scroll down to ‘Other admissions tests and assessments’ where you should click on ‘STEP Mathematics’, then ‘Preparing for STEP’ where you will find ‘STEP past papers’.</p>

Past and specimen examination papers

Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support **(F)**

The 2020 syllabus includes changes to the assessment structure, assessment objective weightings and syllabus content when compared to the 2017–2018 and 2019 syllabuses. Therefore, if you use past papers and mark schemes from earlier series, please do so with caution. It is still possible to help your learners understand what the examination papers look like and to give an idea of the required standard but please be aware that some of the content, the assessment structure and nature of the mark scheme has changed. Please also use the specimen papers and mark schemes for the 2020 series.

Specimen Paper 2 Q2 integration, Q4 Rectangle areas, Question 5 Arc length, surface area, Q7(c) Maclaurin series

2.5 Complex numbers

Syllabus subject content	Suggested teaching activities
Understand de Moivre's theorem, for a positive or negative integer exponent, in terms of the geometrical effect of multiplication and division of complex numbers	<p>Prior knowledge: A reminder of the different ways of writing complex numbers can be given using a Tarsia activity from Mr Barton's Maths website (www.mrbartonmaths.com): click on 'Students', then look under Further Pure 2 and click on 'Further Pure 2 Revision Notes (2016)'.</p> <p>Introduction: De Moivre's theorem for positive integral components can be investigated by drawing some simple examples and exploring what geometrical transformation has taken place after multiplication.</p> <p>Main theme: Learners can work from their basic knowledge of complex number multiplication and see how de Moivre's Theorem works for positive integral components. The notes on the Maths Support website (mathsupport.mas.ncl.ac.uk) are informative. Scroll down to 'Pure Maths' and then click on 'Algebra' and here you will find a list of links under the heading 'Complex numbers'. Also see 'Project Maths: A More Geometric Approach to De Moivre's Theorem'.</p>
Prove de Moivre's theorem for a positive integer exponent, e.g. by induction	<p>Main theme: This proof can be left to learners to attempt for themselves using the polar or exponential form for z. If they've already covered induction, then they could also attempt to prove it by induction. (I)</p> <p>Extension activity: Many learners will be able to work out proofs for both negative and fractional exponents using the positive integer exponent as a starting point.</p>
<p>Use de Moivre's theorem for a positive or negative rational exponent, e.g. expressing $\cos 5\theta$ in terms of $\cos \theta$ or $\tan 5\theta$ in terms of $\tan \theta$</p> <ul style="list-style-type: none"> – to express trigonometrical ratios of multiple angles in terms of powers of trigonometrical ratios of the fundamental angle, e.g. expressing $\sin^6 \theta$ in terms of $\cos 2\theta$, $\cos 4\theta$ and $\cos 6\theta$ – to express powers of $\sin \theta$ and $\cos \theta$ in terms of multiple angles – in the summation of series, e.g. using the 'C + iS' method to sum 	<p>Main theme: It is easy for learners to mix up the methods for the first two bullet points. They need to remember that to express multiple angles in terms of the powers of trigonometrical ratios, it is quicker to expand $(\cos \theta + i \sin \theta)^n$ and pick out the real or imaginary part as required.</p> <p>When learners are asked to express powers of sin and cos in terms of multiple angles, they should be encouraged to use the fact that $2\cos \theta = z + \frac{1}{z}$ and $2i \sin \theta = z - \frac{1}{z}$ rather than the expansion used previously, as it will require a lot less manipulation.</p> <p>(F) This topic (and the previous topics) is covered on the PPLATO: Promoting Physics Learning and Teaching Opportunities website (www.met.reading.ac.uk/pplato2). Scroll down to 'Mathematics' and then to 'MATH 3.3: Complex algebra and Demoivre's theorem'.</p> <p>Finding the nth roots of unity is covered in detail in a video by the 'ukmathsteacher', which can be found on Youtube (www.youtube.com).</p>

Syllabus subject content	Suggested teaching activities
<p>series such as $\sum_{r=1}^n \binom{n}{r} \sin r\theta$</p> <p>– in finding and using the nth roots of unity.</p>	
Past examination papers	
<p>Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support (F)</p> <p>The 2020 syllabus includes changes to the assessment structure, assessment objective weightings and syllabus content when compared to the 2017–2018 and 2019 syllabuses. Therefore, if you use past papers and mark schemes from earlier series, please do so with caution. It is still possible to help your learners understand what the examination papers look like and to give an idea of the required standard but please be aware that some of the content, the assessment structure and nature of the mark scheme has changed. Please also use the specimen papers and mark schemes for the 2020 series.</p> <p>Specimen Paper 2 Q6 de Moivre, roots of equation</p>	

2.2 Matrices

Syllabus subject content	Suggested teaching activities
<p>Formulate a problem involving the solution of 3 linear simultaneous equations in 3 unknowns as a problem involving the solution of a matrix equation, or vice versa</p> <p>Understand the cases that may arise concerning the consistency or inconsistency of 3 linear simultaneous equations, relate them to the singularity or otherwise of the corresponding matrix, solve consistent systems, and interpret geometrically in terms of lines and planes, e.g. three planes meeting in a common point, or in a common line, or having no common points.</p>	<p>Prior knowledge: Learners should have already covered the background to matrices in Further Pure Mathematics 1 (for Paper 1), and be able to work out the inverse of a 3×3 matrix. They should also have identified when the matrix is singular and when it is non-singular.</p> <p>Main Theme: There are notes on this topic on the Mathedup! website (www.mathedup.co.uk). From the home page, use the search tool to search for 'solving linear equations'. The notes include worked examples and examples for the learners to solve themselves. (F) There is also an investigation here exploring and illustrating the geometric representations associated with the different cases.</p> <p>More notes, examples and an exercise can be found in Unit 9.5 of Just the Maths booklets. 'Just the Maths' is a collection of units written by A J Hobson that is available online (https://archive.uea.ac.uk/jtm). (F)</p> <p>There are also useful notes and worked examples on the Integral website (http://integralmaths.org); look under the MEI Further Pure 2 section for matrices. (I)</p>
<p>Understand the terms 'characteristic equation', 'eigenvalue' and 'eigenvector', as applied to square matrices, including use of the definition $\mathbf{Ae} = \lambda \mathbf{e}$ to prove simple properties, e.g. that λ^n is an eigenvalue of \mathbf{A}^n</p> <p>Find eigenvalues and eigenvectors of 2×2 and 3×3 matrices (restricted to cases where the eigenvalues are real and distinct)</p>	<p>Main theme: As well as knowing these technical definitions, it's important that learners are confident in using the fundamental link $\mathbf{Ae} = \lambda \mathbf{e}$. Some examples of this can be found in Unit 9.8 of Just the Maths booklets. Just the Maths booklets. 'Just the Maths' is a collection of units written by A J Hobson that is available online (https://archive.uea.ac.uk/jtm). Please note that the exercises that follow the notes in the 'Just the Maths' booklets are not appropriate here. It's worth emphasising that the eigenvalues of a triangular matrix are the entries on the diagonal – this can save a lot of time!</p> <p>There are various techniques for finding the eigenvectors, but it is important that learners take great care when using the vector product method for the eigenvectors not to make errors. Some self-checking can be done, as eigenvalues will be distinct and real. The geometric significance of the eigenvalue and eigenvector should be explained.</p> <p>There are notes and practice questions on finding eigenvalues and eigenvectors on both Just the Maths and Integral (http://integralmaths.org) websites. (F) There are also some notes on the Mathedup! website (www.mathedup.co.uk): from the home page, use the search tool to search for 'eigenvectors'.</p>

Syllabus subject content	Suggested teaching activities
<p>Express a square matrix in the form \mathbf{QDQ}^{-1}, where \mathbf{D} is a diagonal matrix of eigenvalues and \mathbf{Q} is a matrix whose columns are eigenvectors, and use this expression, e.g. in calculating powers of 2×2 or 3×3 matrices</p>	<p>Main theme: This process is relatively straightforward, though care should be taken as it will only work if the eigenvectors that make up \mathbf{Q} are linearly independent. Care must also be taken to make sure that the eigenvalues and vectors are in corresponding positions. There are notes and examples of this method on the MathedUp! website (www.mathedup.co.uk) under Further Pure 4.</p>
<p>Use the fact that a square matrix satisfies its own characteristic equation, e.g. in finding successive powers of a matrix or finding an inverse matrix; restricted to 2×2 or 3×3 matrices only.</p>	<p>Main theme: Notes on this can be found on the Integral website (http://integralmaths.org) under the section in MEI Further Pure 2 on matrices: look for the Cayley Hamilton Theorem. There are also examples here too. (F) There is also a revision video on the student revision area of FSMP (furthermaths.org.uk): under 'Students' select 'Revision' then 'MEI Revision videos' and under 'Further Pure 2'. The video goes through some examination questions; it also covers worked examples with powers of matrices. (I).</p>
Past examination papers	
<p>Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support (F)</p> <p>The 2020 syllabus includes changes to the assessment structure, assessment objective weightings and syllabus content when compared to the 2017–2018 and 2019 syllabuses. Therefore, if you use past papers and mark schemes from earlier series, please do so with caution. It is still possible to help your learners understand what the examination papers look like and to give an idea of the required standard but please be aware that some of the content, the assessment structure and nature of the mark scheme has changed. Please also use the specimen papers and mark schemes for the 2020 series.</p> <p>Specimen Paper 2 Q8 system of equations, eigenvalues</p>	

2.6 Differential equations

Syllabus subject content	Suggested teaching activities
<p>Find an integrating factor for a first order linear differential equation, and use an integrating factor to find the general solution, e.g.</p> $\frac{dy}{dx} - 2y = x^2, \quad x \frac{dy}{dx} - y = x^4,$ $\frac{dy}{dx} + y \coth x = \cosh x$	<p>Prior knowledge: Learners should have studied simple differential equations and differential equations with separable variables in Cambridge International AS & A Level Mathematics 9709 Pure Mathematics 3.</p> <p>Main theme: Before introducing the concept of integrating factors, learners could explore exact differential equations by looking at differentials of products, for example, $x^3 \frac{dy}{dx} + 3x^2 = \sin x$, and identifying the left hand-side as the derivative of a product. They can then explore the idea of multiplying a differential equation of the form $\frac{dy}{dx} + Py = Q$ by a function of x to make the left-hand side an exact differential. Learners must make sure that the equation is in the correct form before finding the integrating factor.</p> <p>There are notes on this method on the Imperial College of London Metric website (http://wwwf.imperial.ac.uk/metric/metric_public/). METRIC is a bank of mathematics resources. Click on the link for 'Differential equations'. On the Mathedup! website (www.mathedup.co.uk) learners can access a powerpoint running through details of how to find the integrating factor; search for 'differential equations' and scroll down to click on '1st order differential equations' (this is a Further Pure 3 resource). (I) There are also exercises here. (F)</p>
<p>Recall the meaning of the terms 'complementary function' and 'particular integral' in the context of linear differential equations, and recall that the general solution is the sum of the complementary function and a particular integral</p> <p>Find the complementary function for a first or second order linear differential equation with constant coefficients; for second order equations, including the cases where the auxiliary equation has distinct real roots, a repeated real root or conjugate complex roots</p> <p>Recall the form of, and find, a particular integral for a first or second order linear differential equation in the cases where a polynomial or ae^{bx} or $a \cos px + b \sin px$ is a suitable form, and in</p>	<p>Main theme: The language here is important so that learners remember the correct terms for the different parts of the solution. Learners need to know how to choose a suitable form for the particular integral in the three cases required. For a walkthrough of the method in the context of first order equations, the powerpoint '1st order differential equations 2' on the Mathedup! website (www.mathedup.co.uk) gives a straightforward explanation of the method; search for 'differential equations' and then scroll down to click on '1st order differential equations'. There is a test on the topic here too.</p> <p>Learners need to go through the three possibilities for the complementary function with second order equations. Learners need to learn the complementary functions relevant to each of the three possible types of solution to the auxiliary equation. Questions will not always involve x and y, and it's really important that learners use the variables of the questions consistently, not automatically reverting to x and y. Where an appropriate form for the particular integral is given, learners should use it, and not waste time trying to find it for themselves.</p> <p>There are notes, examples and exercises on the PPLATO: Promoting Physics Learning and Teaching Opportunities website (www.met.reading.ac.uk/pplato2), and on the Mathcentre website (www.mathcentre.ac.uk); search for 'first order differential equations with integrating factor' or 'second</p>

Syllabus subject content	Suggested teaching activities
<p>other simple cases find the appropriate coefficient(s) given a suitable form of particular integral, e.g. evaluate k given that $kx \cos 2x$ is a particular integral of $\frac{d^2y}{dx^2} + 4y = \sin 2x$</p>	<p>order differential equations'. The Integral website (http://integralmaths.org) also covers the topic in the OCR Further Pure 3 section. (F)</p> <p>Extension activity: There are some challenging problems on the NRICH website (http://nrich.maths.org) that make learners really think about differential equations in context.</p>
<p>Use a given substitution to reduce a differential equation to a first or second order linear equation with constant coefficients or to a first order equation with separable variables, e.g.</p> <p>the substitution $x = e^t$ to reduce to linear form a differential equation with terms of the form</p> $ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy,$ <p>or the substitution $y = ux$ to reduce</p> $\frac{dy}{dx} = \frac{x+y}{x-y}$ <p>to separable form</p>	<p>Main theme: Substitutions are given, but having 'extra' variables in a question can be confusing, so learners need to take great care to use the correct variables at each stage.</p> <p>A PowerPoint with a worked example for a second order differential equation can be found on the Exams Solutions website (www.examsolutions.net). (I) There are some videos of worked exam questions on the MathedUp! website (www.mathedup.co.uk) that are also helpful for revision; the topic can be found under 'Further Pure 3'. (I)</p>
<p>Use initial conditions to find a particular solution to a differential equation, and interpret a solution in terms of a problem modelled by a differential equation.</p>	<p>Main theme: As well as using the boundary conditions to find the particular solution, learners need to be able to identify the limits of the solution and what the solution represents in terms of the problem. Some practice can be found on the Maths Support website (http://mathsupport.mas.ncl.ac.uk/mediawiki/Main_Page); 'Math Support' is a joint project between Newcastle University and the University of Birmingham. Other practice materials can be found on the PPLATO website (www.met.reading.ac.uk/pplato2).</p> <p>Extension activity: There are some challenging problems on the NRICH (http://nrich.maths.org) that make learners really think about differential equations in context. Search for 'differential equations' and then select, for example, 'Advanced problem solving module 17'.</p>

Past examination papers

Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support **(F)**

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Specimen Paper FP2 Q1 second order, Q3 integrating factor

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