

Scheme of Work

Cambridge International AS & A Level Further Mathematics 9231

Further Pure Mathematics 1 (for Paper 1)

For examination from 2020



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Introduction

This scheme of work has been designed to support you in your teaching and lesson planning. The scheme of work has been separated into four documents, one for each examination Paper: Further Pure Mathematics 1 (for Paper 1); Further Pure Mathematics 2 (for Paper 2); Further Mechanics (for Paper 3); and Further Probability & Statistics (for Paper 4). This document relates only to **Further Pure Mathematics 1 (for Paper 1)**.

Making full use of this scheme of work will help you to improve both your teaching and your learners' potential. It is important to have a scheme of work in place in order for you to guarantee that the syllabus is covered fully. You can choose what approach to take and you know the nature of your institution and the levels of ability of your learners. What follows is just one possible approach you could take and you should always check the syllabus for the content of your course.

There is a separate table for each topic of the Further Pure Mathematics 1 syllabus content (1.1 Roots of polynomial equations, 1.2 Rational functions and graphs, etc.). Each of the bullet points from the syllabus subject content are listed along with teaching suggestions. There is a 'Main theme' for each, which is the focus activity/activities for the content. Where possible, this is supported by an 'Introduction' activity to set the context. Suggestions for independent study (**I**) and formative assessment (**F**) are also included. Opportunities for differentiation are indicated as **Extension activities**; there is the potential for differentiation by resource, grouping, expected level of outcome, and degree of support by teacher, throughout the scheme of work. Timings for activities and feedback are left to the judgment of the teacher, according to the level of the learners and size of the class. Length of time allocated to a task is another possible area for differentiation.

Key concepts

This scheme of work is underpinned by the assumption that Mathematics is fundamentally problem solving and representing systems and models in different ways. The key concepts are highlighted as a separate item in the new syllabus and teachers should be aware that learners will be assessed on their direct knowledge and understanding of the same. Learners should be able to describe and explain the key concepts as well as demonstrate their ability to apply them to novel situations and evaluate them. They are not referred to specifically in the Scheme of Work as they are essential to tackling problems in all topics.

The key concepts are as follows:

- **Problem solving**

Mathematics is fundamentally problem solving and representing systems and models in different ways.

These include:

- Algebra: this is an essential tool which supports and expresses mathematical reasoning and provides a means to generalise across a number of contexts.
- Geometrical techniques: algebraic representations also describe a spatial relationship, which gives us a new way to understand a situation.
- Calculus: this is a fundamental element which describes change in dynamic situations and underlines the links between functions and graphs.
- Mechanical models: these explain and predict how particles and objects move or remain stable under the influence of forces.
- Statistical methods: these are used to quantify and model aspects of the world around us. Probability theory predicts how chance events might proceed, and whether assumptions about chance are justified by evidence.

- **Communication**

Mathematical proof and reasoning is expressed using algebra and notation so that others can follow each line of reasoning and confirm its completeness and accuracy. Mathematical notation is universal. Each solution is structured, but proof and problem solving also invite creative and original thinking.

- **Mathematical modelling**

Mathematical modelling can be applied to many different situations and problems, leading to predictions and solutions. A variety of mathematical content areas and techniques may be required to create the model. Once the model has been created and applied, the results can be interpreted to give predictions and information about the real world.

Recommended prior knowledge

Knowledge of the Cambridge IGCSE® Mathematics 0580 syllabus (or equivalent) is required for the Cambridge International AS & A Level Further Mathematics 9231 course. All topics from the Cambridge International AS & A Level Mathematics 9709 course are also needed as prior knowledge. However, it is possible to teach both A Level courses alongside each other in parallel, as not all of the Further Mathematics topics have direct dependencies. See *Parallel teaching – A two-year plan to co-teach Cambridge International AS & A Level Mathematics 9709 and Cambridge International AS & A Level Further Mathematics 9231* for guidance. This is available on the School Support Hub www.cambridgeinternational.org/support

Guided learning hours

Guided learning hours give an indication of the amount of contact time teachers need to have with learners to deliver a particular course. Our syllabuses are designed around 180 hours for Cambridge International AS Level, and 360 hours for Cambridge International A Level. The number of hours will vary depending on local practice and your learners' previous experience of the subject.

It is recommended that you spend about 90 hours teaching the content for each Paper: Further Pure Mathematics 1 (for Paper 1); Further Pure Mathematics 2 (for Paper 2); Further Mechanics (for Paper 3); and Further Probability & Statistics (for Paper 4).

The table below gives some guidance about how many hours are recommended for each topic within Further Pure Mathematics 1 (for Paper 1).

Syllabus content topic	Suggested teaching time (hours)
1.7 Proof by induction	15
1.3 Summation of series	10
1.1 Roots of polynomial equations	10
1.2 Rational functions and graphs	10
1.5 Polar coordinates	15
1.6 Vectors	15
1.4 Matrices	15

Resources

The textbooks endorsed by Cambridge International for use with this course are listed at www.cambridgeinternational.org. Endorsed textbooks have been written to be closely aligned to the syllabus they support, and have been through a detailed quality assurance process. As such, all textbooks endorsed by Cambridge International for this syllabus are the ideal resource to be used alongside this scheme of work as they cover each learning objective. There is also a support resource available for co-teaching the Cambridge International AS & A Level Further Mathematics 9231 course alongside the Cambridge International AS & A Level Mathematics 9709 course: *Parallel teaching – A two-year plan to co-teach Cambridge International AS & A Level Mathematics (9709) and Cambridge International AS & A Level Further Mathematics 9231*, which is available on the School Support Hub.

School Support Hub

The School Support Hub www.cambridgeinternational.org/support is a secure online resource bank and community forum for Cambridge teachers, where you can download specimen and past question papers, mark schemes and other resources. We also offer online and face-to-face training; details of forthcoming training opportunities are posted online. This scheme of work is available as PDF and an editable version in Microsoft Word format from the School Support Hub. If you are unable to use Microsoft Word you can download Open Office free of charge from www.openoffice.org.

Websites

This scheme of work includes website links providing access to internet resources. Cambridge Assessment International Education is not responsible for the accuracy or content of information contained in these sites. The inclusion of a link to an external website should not be understood to be an endorsement of that website or the site's owners (or their products/services).

The website pages referenced in this scheme of work were selected when the scheme of work was produced. Other aspects of the sites were not checked and only particular resources are recommended.

Useful websites include:

www.stem.org.uk The National STEM Learning Network provides access to a range of resources.

<http://integralmaths.org> The Integral® website provides resources developed by a curriculum development project called MEI. Since these schemes were first written, this website has become available only through paid subscription.

www.mmlsoft.com/index.php/products/tarsia The TARSIA software is free to download. It can be used to download and create puzzles to test manipulation.

Important notice about past papers

The 2020 syllabus (for examination in 2020) includes changes to the assessment structure, assessment objective weightings and syllabus content when compared to the 2017–2018 and 2019 syllabuses. Therefore, if you use past papers and mark schemes from earlier series, please do so with caution. It is still possible to help your learners understand what the examination papers look like and to give an idea of the required standard but please be aware that some of the content, the assessment structure and nature of the mark scheme has changed. Please also use the Specimen Papers and mark schemes for the 2020 series.

How to get the most out of this scheme of work – integrating syllabus content, skills and teaching strategies

We have written this scheme of work for the Cambridge International AS & A Level Further Mathematics 9231 syllabus and it provides some ideas and suggestions of how to cover the content of the syllabus. We have designed the following features to help guide you through your course.

Syllabus subject content lists the subject content bullet points from the syllabus, making it clear the knowledge your learners need to build. Pass these on to your learners by expressing them as 'We are learning to / about...'.
(I)

Suggested teaching activities give you lots of ideas about how you can present learners with new information. Try more active methods that get your learners motivated and practising new skills. Where possible, the activities are separated into 'Introduction' ideas to set the context, and 'Main themes' that form the core of the teaching.
(I)

Syllabus subject content

use the method of mathematical induction to establish a given result;

Suggested teaching activities

Introduction: To put this topic in context, give some examples of deductive proofs, such as proving the formula for the sum of an algebraic or geometric series; or the quadratic formula. It is important to make sure learners understand the need to show mathematical rigour at every step. (I)

Extension activity: Proof by contradiction and by exhaustion could also be examined as an extension activity from the same resource. Some classical proofs are illustrated on <https://undergroundmathematics.org>, search for 'Divisibility & Induction'.

Main theme: Proving familiar results such as the sum of the positive integers is a good place to start. Some useful examples and exercises can be found on the Integral website (<http://integralmaths.org>). There is a good matching activity on the STEM website (www.stem.org.uk) called 'Creating Connections Between Topics: Proof by Induction'. (F)

Divisibility tests, inequalities, calculus, geometry and series may all be used as contexts and it's always important that the deductive step is written out fully with a rigorous argument. Note: some questions involving calculus may require techniques from Cambridge International AS & A Level Mathematics (9709) Pure Mathematics 3, so take this into consideration when planning.

Extension activities provide your more able learners with further challenge beyond the basic content of the course. Innovation and independent learning are the basis of these activities.
(I)

Independent study (I) gives your learners the opportunity to develop their own ideas and understanding with direct input from you.
(I)

Past and specimen papers

Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support (F)
Specimen Paper 1 Q2 Divisibility

Past papers, specimen papers and mark schemes are available for you to download at:
www.cambridgeinternational.org/support

Using these resources with your learners allows you to check their progress and give them confidence and understanding.

Formative assessment (F) is on-going assessment that informs you about the progress of your learners. Don't forget to leave time to review what your learners have learnt, you could try question and answer, tests, quizzes, 'mind maps', or 'concept maps'.

1.7 Proof by induction

Syllabus subject content	Suggested teaching activities
<p>Use the method of mathematical induction to establish a given result; e.g.</p> $\sum_{r=1}^n r^4 = \frac{1}{4}n^2(n+1)^2,$ $u_n = \frac{1}{2}(1+3^{n-1}) \text{ for the sequence given by}$ $u_{n+1} = 3u_n - 1 \text{ and } u_1 = 1,$ $\begin{pmatrix} 4 & -1 \\ 6 & -1 \end{pmatrix}^n = \begin{pmatrix} 3 \times 2^n - 2 & 1 - 2^n \\ 3 \times 2^{n+1} - 6 & 3 - 2^{n+1} \end{pmatrix},$ $3^{2n} + 2 \times 5^n - 3 \text{ is divisible by } 8.$ <p>Recognise situations where conjecture based on a limited trial followed by inductive proof is a useful strategy, and carry this out in simple cases,</p> <p>e.g. find the nth derivative of xe^x, find $\sum_{r=1}^n r \times r!$</p>	<p>Prior knowledge: The level of algebraic handling can be quite high. Knowledge of geometry and matrices from Cambridge IGCSE or equivalent might be required for some proofs. This topic may involve calculus from Pure Mathematics 1 and 3, from Cambridge International AS & A Level Mathematics (9709).</p> <p>Introduction: To put this topic in context, give some examples of deductive proofs, such as proving the formula for the sum of an algebraic or geometric series; or the quadratic formula. It is important to make sure learners understand the need to show mathematical rigour at every step. Some false proofs can be examined to test these ideas further, and learners should be encouraged to spot the mistakes in the reasoning. Examples of these can be found on the NRICH website (http://nrich.maths.org) by searching for ‘STEP Activities for Proof’ (I).</p> <p>Extension activity: Proof by contradiction and by exhaustion could also be examined as an extension activity from the same resource. Some classical proofs are illustrated on https://undergroundmathematics.org, search for ‘Divisibility & Induction’.</p> <p>Main theme: The structure of an induction proof needs careful explanation. There is a simple example on the Khan Academy website (www.khanacademy.org) that gives a good idea of the structure. Its title is ‘Proof by Induction’. Using the analogy of climbing steps helps – if you can get on the first step and move from one step to the next, then you can get to the top. Showing that if a proposition is true for some arbitrary value of n, say k, then it is also true for the case where $n = k + 1$, can be compared to getting from any step to the next step. Proving the proposition is true for (usually) $n = 1$ is essential to get started on the journey up. The language of an induction proof is very important and learners should practise using it in full, avoiding use of statements like ‘assume $n = k$’ as short cuts.</p> <p>Proving familiar results such as the sum of the positive integers is a good place to start. Some useful examples and exercises can be found on the Integral website (http://integralmaths.org). There is a good matching activity on the STEM website (www.stem.org.uk) called ‘Creating Connections Between Topics: Proof by Induction’. (F)</p> <p>Divisibility tests, inequalities, calculus, geometry and series may all be used as contexts and it’s always important that the deductive step is written out fully with a rigorous argument. Note: some questions involving calculus may require techniques from Cambridge International AS & A Level Mathematics (9709) Pure Mathematics 3, so take this into consideration when planning.</p>

Past and specimen papers

Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support **(F)**

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Specimen Paper 1 Q2 divisibility

1.3 Summation of series

Syllabus subject content	Suggested teaching activities
Use the standard results for $\sum r$, $\sum r^2$, $\sum r^3$ to find related sums.	<p>Prior knowledge: Learners need to be familiar with the sigma notation; this is covered in geometric and arithmetic series in Cambridge International AS & A Level Mathematics (9709) Pure Mathematics 1.</p> <p>Introduction: The three formulae could be proved by induction, if the syllabus content '1.7 Proof by induction' has already been covered.</p> <p>Main theme: These types of problems depend on algebraic manipulation to reduce complex expressions to one or more of the standard formulae. Learners can often find a way to approach a summation themselves, making this a possible learner-led activity.</p> <p>Discussion of different methods followed by a summary of successful approaches should cover all the necessary types. Setting a variety of summations needing the expansion of brackets; subtraction of $\sum_{r=1}^n u_r$ from $\sum_{r=1}^{2n} u_r$; and manipulation of odd and even terms prepares learners for the range of techniques needed.</p> <p>Although learners might remember the three formulae, they should check the list of formulae and statistical tables in the Notation List in an examination, rather than relying on memory.</p> <p>There are worksheets on the Integral site (http://integralmaths.org) that give extra practice at different levels. For example, search for the 'Summation of Series' examples for OCR Further Pure 1. (F)</p> <p>Extension activity: There are some interesting and challenging examples on the NRICH website (http://nrich.maths.org): search for 'Summation of Series' and filter for Stage 5 to get problems at the highest level, such as 'Weekly Challenge 36: Seriesly'; 'Generally Geometric'; 'Telescoping Series' and many other interesting examples.</p>

Syllabus subject content	Suggested teaching activities
<p>Use the method of differences to obtain the sum of a finite series.</p> <p>Note: Use of partial fractions to express a general term in a suitable form may be required.</p>	<p>Prior knowledge: The process of splitting an algebraic fraction into partial fractions is covered in Cambridge International AS & A Level Mathematics (9709) Pure Mathematics 3 and is core to many of the problems. The Mr Barton's Maths website (www.mrbartonmaths.com/teachers.htm) has some Tarsia domino and jigsaw-type puzzles that practise these in a slightly different way to a textbook exercise. You will need to download the free Tarsia software (www.mm1soft.com/index.php/products/tarsia) before being able to do the activities.</p> <p>Main theme: Show learners how to set out their work so that any cancelling patterns are easy to spot. Encourage them to write down at least three or four terms from each end of the summation, to fully establish which terms will remain after cancelling. If candidates are asked to show that an expression can be written in partial fractions in the examination, a detailed working is needed.</p> <p>(F) Sometimes learners will need to find the nth term of a sequence by using the difference between successive sums. There are examples and exercises on the Integral website (http://integralmaths.org), and on the Maths at Hawthorn website (http://mathsathawthorn.pbworks.com) – on the latter, look under 'Further Pure 1' in the 'Navigator' panel.</p>
<p>Recognise, by direct consideration of a sum to n terms, when a series is convergent, and find the sum to infinity in such cases.</p>	<p>Prior knowledge: Learners should be familiar with the sum of an infinite geometric progression.</p> <p>Main theme: The technique studied here is to investigate the sum of n terms, which is then used to determine whether a series is convergent. It is also used to find the sum to infinity if it exists, so the sum of a geometric progression makes a useful example.</p> <p>(I) Learners might find the videos on Further Mathematics Support Programme (www.furthermaths.org.uk) helpful. Click on the 'Students' tab and then select 'Revision' to reveal a list of links. Click on 'OCR revision videos' to provide a list of videos; the 'Summation of series' video covers this topic, as well as the previous elements on series. The Mathcentre website (www.mathcentre.ac.uk) also has comprehensive notes: search for 'Sum of an Infinite Series'.</p> <p>Extension activity: Ask learners to investigate the formulae $\sum \frac{1}{(k-1)!}$ to determine what happens when they add more and more terms. The series for e and π are also explained on the mathcentre website (www.mathcentre.ac.uk), though some of their material uses a different notation. There are plenty of interesting series to investigate on the NRIC website (http://nrich.maths.org), for example 'Telescoping Series' or 'Reciprocal Triangles'.</p>

Past and specimen papers

Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support (F)

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Specimen Paper 1 Q1 difference method with partial fractions, sum to infinity

1.1 Roots of polynomial equations

Syllabus subject content	Suggested teaching activities
<p>Recall and use the relations between the roots and coefficients of polynomial equations,</p> <p>e.g. to evaluate symmetric functions of the roots or to solve problems involving unknown coefficients in equations; restricted to equations of degree 2, 3 or 4 only.</p>	<p>Prior knowledge: Learners should be able to solve quadratic equations by factorising to identify integer roots.</p> <p>Main theme: Learners can investigate the connection between the roots of a quadratic equation and the coefficients, and then justify their findings by multiplying out the linear factors and matching coefficients. It's easier to start with the coefficient of $x^2 = 1$. This can then be extended to other coefficients of x^2, and to cubics and quartics.</p> <p>The simple identity $\sum \alpha^2 \equiv (\sum \alpha)^2 - 2\sum \alpha\beta$ is worth learning, but the more complicated formulae are sometimes misquoted, so it's probably better to use the equations themselves for higher powers.</p> <p>Learners sometimes forget that the original equation holds for each root, often giving a useful additional relationship. Equations can also be multiplied by powers of x giving additional relationships.</p> <p>The video 'Coefficients and Roots of Polynomials 1' from StPaulsmaths2012 is a useful resource (www.youtube.com/watch?v=JdvnyjJRZ_8).</p> <p>The Integral website (http://integralmaths.org) has some notes and examples for polynomials up to degree 3. Look at 'MEI Further Pure 1', which goes up to quartic equations or 'OCR Further Pure 1' which could be used for independent study (I) or revision up to cubic equations (F).</p> <p>(F) There are also notes, with short exercises, on this topic on the Community Resources area of School Support Hub (www.cambridgeinternational.org/support).</p> <p>Extension activity: Learners can investigate the relations for polynomials beyond a quartic, and try to find a general rule.</p>

Syllabus subject content	Suggested teaching activities
<p>Use a substitution to obtain an equation whose roots are related in a simple way to those of the original equation.</p> <p>Note: Substitutions will not be given for the easiest cases, e.g. where the new roots are reciprocals or squares or a simple linear function of the old roots.</p>	<p>Main theme: This follows on from the previous section. PowerPoint files supporting the process of substitution can be found on the Integral website (https://integralmaths.org) as above.</p> <p>If a substitution is given or the substitution method is specified in the question, then using a substitution will be the quickest way to the answer. If no mention is made of substitution, then sometimes the methods of the previous section, i.e. using the original equation and manipulating the sums and products of the roots, can be relatively straightforward.</p> <p>(F) There are notes, with short exercises, on this topic on the Community Resources area of School Support Hub (www.cambridgeinternational.org/support).</p>
Past and specimen papers	
<p>Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support (F)</p> <p>The 2020 syllabus includes changes to the assessment structure, assessment objective weightings and syllabus content when compared to the 2017–2018 and 2019 syllabuses. Therefore, if you use past papers and mark schemes from earlier series, please do so with caution. It is still possible to help your learners understand what the examination papers look like and to give an idea of the required standard but please be aware that some of the content, the assessment structure and nature of the mark scheme has changed. Please also use the Specimen Papers and mark schemes for the 2020 series.</p> <p>Specimen Paper 1 Q4 roots, related cubic</p>	

1.2 Rational functions and graphs

Syllabus subject content	Suggested teaching activities
<p>Sketch graphs of simple rational functions, including the determination of oblique asymptotes, in cases where the degree of the numerator and the denominator are at most 2.</p> <p>Note: Includes determination of the set of values taken by the function, e.g. by the use of a discriminant. Detailed plotting of curves will not be required, but sketches will generally be expected to show significant features, such as turning points, asymptotes and intersections with the axes.</p>	<p>Prior knowledge: Learners will need to have covered differentiation in their Cambridge International AS & A Level Mathematics (9709) course.</p> <p>(F) Learners also need to be able to divide polynomials, or find the quotient and remainder by equating coefficients. They can practise this using a card matching activity from a wide collection available on the Mr Barton's Maths website (www.mrbartonmaths.com). You need to download the Tarsia free software before you can use the Tarsia activities (www.mmlsoft.com/index.php/products/tarsia). Search for 'Tarsia Jigsaw Puzzles' on the site and select 'Tarsia Jigsaws on Mr Barton Maths'. Scroll down the page until you reach the heading 'Mr Barton's Tarsia Jigsaw Files', then select 'Core 4' from the list of links. The 'Algebraic division' activities you need are in the folder called '1. Algebra'. Notes on polynomial division can also be found in the Algebra section of the mathtutor website (www.mathtutor.ac.uk) along with practice exercises.</p> <p>Main theme: Although learners must be able to sketch without a graphing package, it is a good idea to use either a graphic calculator or a graphing package to inform and confirm their efforts in the early stages. Geogebra (www.geogebra.org/download) is an example of suitable free online graphing software. It is possible to explore graphs and identify the equations of the vertical asymptotes and then come back to look at the algebra. The polynomial division needs to be carried through completely so that the remainder is a proper (algebraic) fraction, otherwise the oblique asymptote will be wrong. The key features of the graphs need to be clearly labelled, especially the coordinates of the intercepts with the axes, the turning points and the trends as x and y tend to infinity. These features are not always accurate enough on graphing packages.</p> <p>Extension activity: A good open-ended task that requires learners to work out the equation for a given graph can be found on the Underground Mathematics website (https://undergroundmathematics.org). Search for 'Can you find ... asymptote edition'.</p> <p>(I) Learners also need to find the range of a function which is restricted. This is usually done by rearranging the equation into a quadratic and finding where it doesn't have real roots. For a video that goes through these steps, search YouTube for 'Determining Range in Real Rational Functions' (www.youtube.com/watch?v=cqUW25bzOro). There are also examples on the analyzemath website (www.analyzemath.com). Search for 'Find Range of Rational Functions'.</p>

Syllabus subject content	Suggested teaching activities
<p>Understand and use relationships between the graphs of $y = f(x)$, $y^2 = f(x)$, $y = \frac{1}{f(x)}$, $y = f(x)$ and $y = f(x)$.</p> <p>Note: Including use of such sketch graphs in the course of solving equations or inequalities.</p>	<p>Prior Knowledge: Learners should have a sound knowledge of the graphs of all the basic functions, as well as experience of working with the modulus function. They should know how to change the function to give simple transformations of the graph from their work in Cambridge International AS & A Level Mathematics (9709). There is a card sorting activity on Underground Mathematics (www.undergroundmathematics.org) called 'Transformers' that can be used to practise these. (F)</p> <p>Main theme: Much of this topic can be developed through investigation using graphing software or a graphing calculator. Learners can explore the visual connection between the related graphs, and then justify the connections using their background knowledge. GeoGebra (www.geogebra.org) applets have been set up for this and don't require you or your learners to have prior knowledge of the package, nor for the package to be downloaded. There are several activities including: 'Effect of performing the absolute value upon a function', which allows learners to explore the modulus of a function; and the 'Reciprocal of a Graph' which allows learners to explore the reciprocal of a function. Exploring graphs of the form $y^2 = f(x)$ can be supported by using the activity 'Graph of y squared equal to a function of x'. Simply access the website and search for the title of the activity.</p> <p>Another useful online tool is desmos (www.desmos.com/).</p>
Past and specimen papers	
<p>Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support (F)</p> <p>The 2020 syllabus includes changes to the assessment structure, assessment objective weightings and syllabus content when compared to the 2017–2018 and 2019 syllabuses. Therefore, if you use past papers and mark schemes from earlier series, please do so with caution. It is still possible to help your learners understand what the examination papers look like and to give an idea of the required standard but please be aware that some of the content, the assessment structure and nature of the mark scheme has changed. Please also use the Specimen Papers and mark schemes for the 2020 series.</p> <p>Specimen Paper Q7 asymptotes, range, sketch, modulus graph, inequality</p>	

1.5 Polar coordinates

Syllabus subject content	Suggested teaching activities
<p>Understand the relations between Cartesian and polar coordinates, and convert equations of curves from Cartesian to polar form and <i>vice versa</i>.</p> <p>Note: The convention $r \geq 0$ will be used.</p>	<p>Prior knowledge: Learners need to be familiar with Cartesian coordinates and using radians for angle measure.</p> <p>Main theme: A discussion about ways to identify the position of point on a plane can lead into the method of fixing position by giving an angle (measured from the initial line), and a distance (from the pole). Learners should be able to work out how to convert between the two with some supported investigation. Lots of practice is needed with the conversions, emphasising the use of $x = r \cos \theta$, $y = r \sin \theta$, and $r^2 = x^2 + y^2$. There are notes on the whole topic on the Interactive Mathematics website (www.intmath.com) under 'Plane Analytical Geometry'.</p> <p>It's important to emphasise the convention that $r \geq 0$ is used on this syllabus – this is not always the case with online resources and older textbooks.</p>
<p>Sketch simple polar curves, for $0 \leq \theta < 2\pi$ or $-\pi < \theta \leq \pi$ or a subset of either of these intervals.</p> <p>Note: Detailed plotting of curves will not be required, but sketches will generally be expected to show significant features, such as symmetry, coordinates of intersections with the initial line, the form of the curve at the pole and least/greatest values of r.</p>	<p>Main theme: Whilst learners need to be able to plot curves by hand, they can relatively quickly gain valuable familiarity of a range of polar curves by investigating them on a graphic calculator, or using a graphing package.</p> <p>It's very helpful if they recognise some of the common graphs, but they shouldn't try to learn them all, and need to be prepared for the unfamiliar.</p> <p>Drawing up a table of values and then making sure that key points are clearly marked on the sketch graphs is important. If learners look for symmetries in trigonometric functions particularly, they can save valuable time in the plotting.</p> <p>(I) Learners can access a range of ready-made applets on GeoGebra (www.geogebra.org) by searching for 'polar graphs', giving them the opportunity to explore. Interactive Mathematics (www.intmath.com) also provides a number of sketches; search for 'Curves in Polar Coordinates', under Plane Analytical Geometry. Their examples don't follow the convention that $r \geq 0$, so learners need to work out which part of the curve would be lost.</p>

Syllabus subject content	Suggested teaching activities
<p>Recall the formula $\frac{1}{2} \int r^2 d\theta$ for the area of a sector, and use this formula in simple cases.</p>	<p>Main theme: The formula can be derived by looking at the area bounded by the curve and two radii at θ and $\theta + \delta\theta$, and finding an expression for $\frac{dA}{d\theta}$. Make sure learners know that they need to learn the formula!</p> <p>The integrations often involve trigonometric expressions, so it's good to go over the integration of, for example, $\cos^2\theta$ and $\sin^2\theta$, but also make learners aware that the integration may require integration techniques from Cambridge International AS & A Level Mathematics (9709) Pure Mathematics 3.</p> <p>There are notes on the whole topic in the Community Resources area on the School Support Hub (www.cambridgeinternational.org/support).</p> <p>(F) The topic is also covered on the Integral website (http://integralmaths.org) – search in the MEI Further Pure 2 section – and on the Maths at Hawthorn website (mathsathawthorn.pbworks.com). More notes and examples can be found on the 17 calculus website (17calculus.com).</p>
Past and specimen papers	
<p>Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support (F)</p> <p>The 2020 syllabus includes changes to the assessment structure, assessment objective weightings and syllabus content when compared to the 2017–2018 and 2019 syllabuses. Therefore, if you use past papers and mark schemes from earlier series, please do so with caution. It is still possible to help your learners understand what the examination papers look like and to give an idea of the required standard but please be aware that some of the content, the assessment structure and nature of the mark scheme has changed. Please also use the Specimen Papers and mark schemes for the 2020 series.</p> <p>Specimen Paper 1 Q3 sketch, area, polar-Cartesian</p>	

1.6 Vectors

Syllabus subject content	Suggested teaching activities
Use the equation of a plane in any of the forms $ax + by + cz = d$ or $\mathbf{r} \cdot \mathbf{n} = p$ or $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$ and convert equations of planes from one form to another as necessary in solving problems.	<p>Prior knowledge: Basic knowledge of vector notation, the scalar product and different forms of straight line equations is assumed from Cambridge International AS & A Level Mathematics (9709) Pure Mathematics 3.</p> <p>Main theme: (F) There are some useful notes on vector equations of planes in <i>Just the Maths</i>, which is a collection of units written by A J Hobson that is available online (https://archive.uea.ac.uk/jtm); look for Unit 8.6. Notes and exercises are also available on the Integral site (http://integralmaths.org) – search for ‘vectors’ in the MEI Further Pure 3, or OCR Further Pure 3 sections.</p>
Recall that the vector product $\mathbf{a} \times \mathbf{b}$ of two vectors can be expressed either as $ \mathbf{a} \mathbf{b} \sin \theta \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector, or in component form as $(a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$	<p>Main theme: The definition and execution of the vector product are very important. There are different ways of carrying it out, and learners need to be confident and accurate with whichever method is chosen. Sign errors are common.</p> <p>(F) On each of the websites listed below, search for ‘vectors’ to find useful:</p> <ul style="list-style-type: none"> • videos – mathcentre (www.mathcentre.ac.uk) • PowerPoint files – MathedUp! (www.mathedup.co.uk) a • notes – Just the maths (archive.uea.ac.uk/jtm)
Use equations of lines and planes, together with scalar and vector products where appropriate, to solve problems concerning distances, angles and intersections, including <ul style="list-style-type: none"> – determining whether a line lies in a plane, is parallel to a plane or intersects a plane, and finding the point of intersection of a line and a plane when it exists, – finding the foot of the perpendicular from a point to a plane, – finding the angle between a line and a plane, and the angle between two planes, – finding an equation for the line of intersection of two planes, 	<p>Main theme: This section of the syllabus is problem solving using the properties above, but it’s worth going through examples of these standard elements so that learners don’t have to go back to first principles every time.</p> <p>(F) ‘The Distance Dominoes’ and the ‘Line Groupings’ in the Active Learning section of Integral’s (http://integralmaths.org) MEI Further Pure 3 section on vectors, provide good practise that is more interesting than typical textbook exercises.</p> <p>(F) Although it’s difficult to sketch well in three dimensions, a basic representation of the problem to be solved often helps. Sections 8.5 and 8.6 on the Just the Maths website (archive.uea.ac.uk/jtm) cover these techniques and have some examples.</p> <p>(F) It is also worth looking at the Integral website (http://integralmaths.org) for notes and questions – vector methods are covered in OCR Further Pure 3 for example.</p> <p>(F) Although practising the individual methods is helpful, because of the problem-solving nature of this module, it’s worth putting together worksheets of past examination questions to give learners the chance to select appropriate methods.</p>

Syllabus subject content	Suggested teaching activities
<ul style="list-style-type: none"> – calculating the shortest distance between two skew lines, – finding an equation for the common perpendicular to two skew lines. 	<p>Some learners will prefer to learn formulae for individual techniques, others will prefer to go back to basics each time – the back to basics approach is very robust! None of the standard formulae are in the Notation List.</p> <p>Extension activity: Challenging problems can be found on the NRICH website (http://nrich.maths.org): the activity called 'Walls' is an open-ended investigation of perpendicular planes; and the 'Tetra Perp' activity looks into perpendicular edges for a tetrahedron. This website also lists some relevant STEP problems.</p>
Past and specimen papers	
<p>Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support (F)</p> <p>The 2020 syllabus includes changes to the assessment structure, assessment objective weightings and syllabus content when compared to the 2017–2018 and 2019 syllabuses. Therefore, if you use past papers and mark schemes from earlier series, please do so with caution. It is still possible to help your learners understand what the examination papers look like and to give an idea of the required standard but please be aware that some of the content, the assessment structure and nature of the mark scheme has changed. Please also use the Specimen Papers and mark schemes for the 2020 series.</p> <p>Specimen Paper 1 Q6 shortest distance, angle between planes</p>	

1.4 Matrices

Syllabus subject content	Suggested teaching activities
<p>Carry out operations of matrix addition, subtraction and multiplication, and recognise the terms zero matrix and identity (or unit) matrix.</p> <p>Note: Including non-square matrices. Matrices will have at most 3 rows and columns.</p>	<p>Prior knowledge: Although learners may have come across matrices in earlier courses, there is no assumed prior knowledge here. If your learners have not studied matrices before, then a brief description of how they can be used would be helpful. The Khan academy website (www.khanacademy.org) provides a good a description; search for 'Introduction to Matrices'. (I)</p> <p>Main theme: Mastering these simple operations could be given as a flipped learning exercise, with learners researching online to find out how to perform addition, subtraction and multiplication (I) and then lesson time can be spent practising (F). The Maths is Fun website (www.mathsisfun.com) has a simple introduction, search for 'Matrices'; or for a video introduction, learners could use the Khan Academy website (www.khanacademy.org) – matrices are in the precalculus course.</p>
<p>Recall the meaning of the terms 'singular' and 'non-singular' as applied to square matrices and, for 2×2 and 3×3 matrices, evaluate determinants and find inverses of non-singular matrices.</p> <p>Note: The notations $\det \mathbf{M}$ for the determinant of a matrix \mathbf{M}, and \mathbf{I} for the identity matrix, will be used.</p> <p>Understand and use the result, for non-singular matrices, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$</p> <p>Note: Extension to the product of more than two matrices may be required.</p>	<p>Main theme: Learners can find the inverse of a 2×2 matrix, \mathbf{A}, where $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, by letting $\mathbf{A}^{-1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and then letting $\mathbf{AA}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$ and working out e, f, g and h using the resulting simultaneous equations. This is a good activity to prove the formula but it should be made clear that inverse of a 2×2 matrix can be written down directly: $\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.</p> <p>The process of finding the determinant and inverse of a 2×2 or 3×3 matrix is explained on the Hegarty Maths website (http://mathswebsite.com) – select 'A-Level', then 'Further Pure 1' and then 'Matrices'; registration with this website is free. There are also notes on the Integral website (www.integralmaths.org); search in the MEI Further Pure 1 section of the site.</p> <p>It's important to link this section with the work on transformations, so that learners understand the connection between the properties of matrices and the transformations that they represent.</p>

Syllabus subject content	Suggested teaching activities
<p>Understand the use of 2×2 matrices to represent certain geometric transformations in the x-y plane, in particular:</p> <ul style="list-style-type: none"> understand the relationship between the transformations represented by \mathbf{A} and \mathbf{A}^{-1} recognise that the matrix product \mathbf{AB} represents the transformation that results from the transformation represented by \mathbf{B} followed by the transformation represented by \mathbf{A} recall how the area scale factor of a transformation is related to the determinant of the corresponding matrix find the matrix that represents a given transformation or sequence of transformations. <p>Note: Understanding of the terms ‘rotation’, ‘reflection’, ‘enlargement’, ‘stretch’ and ‘shear’ for 2D transformations will be required. Other 2D transformations are not excluded, but no particular knowledge of them is expected.</p>	<p>Prior knowledge: Learners should have confidence in using the language – rotation, reflection, enlargement, stretch and shear – to describe transformations from their earlier study of geometry.</p> <p>Main theme: An extensive investigation of the matrices used to represent basic transformations can be found in the 2009 MEI conference notes on the MEI Innovators in Mathematics Education website (http://mei.org.uk). Search for ‘Conference 2009’, then click through to the conference handouts. Under the heading ‘Session C’ you will find the appropriate content in the link for ‘5. Teaching Topics: Matrices in FP1’.</p> <p>There are also GeoGebra (www.geogebra.org) activities available where learners can explore the effect of changing the matrix. Activities by Tom Button and other authors are available on the GeoGebra site by searching for ‘Matrix Transformations’. (I)</p> <p>Notes on transformations can be found on the Integral website (www.integralmaths.org), or by searching the STEM learning website (www.stem.org.uk) for appropriate resources. Another interactive demonstration can be found on the STEM learning website by searching for ‘AQA FP1 Matrices Transformations’. (F) A useful matching exercise can also to be found on the STEM learning website, search for ‘AQA FP1 Matrices Matchings Exercise’.</p> <p>Drawing is a good way to help learners explore the relationship between the transformations represented by \mathbf{A} and \mathbf{A}^{-1}; and how the transformation represented by \mathbf{AB} relates to \mathbf{A} and \mathbf{B}. They could investigate by drawing how the determinant of a matrix relates to the area scale factor of the associated transformation. However, they will probably find it easier to use a package such as Geogebra (www.geogebra.org) to explore. Existing examples can be found on the Geogebra website and used without needing to install the software. Search for ‘2×2 matrix transformation’.</p> <p>Some practice examination questions can be downloaded from the Physics & Maths Tutor website (http://pmt.physicsandmathstutor.com). Search under ‘Maths’ and then ‘Questions by Topic’. Then select ‘Further Pure 1’; you can find suitable questions in the list for Edexcel on the resulting webpage.</p>

Syllabus subject content	Suggested teaching activities
<p>Understand the meaning of ‘invariant’ as applied to points and lines in the context of transformations represented by 2×2 matrices, and solve simple problems involving invariant points and invariant lines.</p> <p>e.g. to locate the invariant points of the transformation represented by $\begin{pmatrix} 6 & 5 \\ 2 & 3 \end{pmatrix}$, or to find the invariant lines through the origin for $\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$, or to show that any line with gradient 1 is invariant for $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$</p>	<p>Main Theme: There is a very good dynamic illustration on the Integral Website (www.integralmaths.org) under MEI Further Pure 1: the ‘Walkthrough on Invariant Points’ in the section on ‘Matrices and Simultaneous equations’ allows learners to explore the concept behind this topic. There are comprehensive notes and exercises here too.</p> <p>There are also detailed notes with exercises on this and previous matrix topics, on the Centre for Innovation in Mathematics Teaching (CMT) website (www.cimt.org.uk), available through the link for the ‘Mathematics Enhancement Programme (MEP)’. A more direct link is http://www.cimt.org.uk/projects/mepres/alevel/alevel.htm</p> <p>Learners can explore this syllabus content through Geogebra (www.geogebra.org) by searching for applets on ‘Invariant Lines of a Matrix’ or ‘Transformation of Lines’. In both, the matrix can be changed by moving the points A and B, and the invariant lines can be highlighted.</p>
Past and specimen papers	
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Cambridge Assessment International Education
1 Hills Road, Cambridge, CB1 2EU, United Kingdom
t: +44 1223 553554 f: +44 1223 553558
e: info@cambridgeinternational.org www.cambridgeinternational.org

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