Cambridge Assessment International Education

Example Candidate Responses – Paper 2

Cambridge International AS & A Level Further Mathematics 9231

For examination from 2022

Cambridge Pathway

Contents

Introduction

The main aim of this booklet is to exemplify standards for those teaching Cambridge International AS & A Level Further Mathematics 9231 and to show how different levels of candidates' performance relate to the subject's curriculum and assessment objectives.

In this booklet, candidate responses have been chosen from the June 2022 series to exemplify a range of answers for all the questions on the question paper.

For each question, the response is annotated with a clear explanation of where and why marks were awarded or omitted. In this way, it is possible for you to understand what candidates have done to gain their marks and what they could do to improve their answers.

This document provides illustrative examples of candidate work with examiner commentary. These help teachers to assess the standard required to achieve marks beyond the guidance of the mark scheme. Please also refer to the June 2022 Examiner Report for further detail and guidance.

The questions and mark schemes used here are available to download from the School Support Hub. These files are:

9231 June 2022 Question Paper 23

9231 June 2022 Mark Scheme 23

Past exam resources and other teaching and learning resources are available on the School Support Hub:

www.cambridgeinternational.org/support

How to use this booklet

This booklet goes through the paper one question at a time. The candidate answers are set in a table. In the left-hand column are the candidate answers, and in the right-hand column are the examiner comments.

Example Candidate Response – 1	Examiner comments
2 (a) Find the coefficient of x^2 in the Maclaurin's series for $-\ln\cos x$. $\cancel{f(x)}$: $f(0) = 0$ $f'(x) = -\frac{1}{\cos x} \cdot -\sin x$ $= \tan x$ (1) $f'(0) = 0$ $f''(x) = \sec^2 x$ (2) $f''(0) = 1$ $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0)$ $= 0 + 0 + 1 \cdot \frac{x^2}{2}$ $= \frac{1}{2}x^2$ (3)	(1) The first derivative seen here is correct and most candidates could derive this using their prior knowledge from 9709. (2) The second derivative is also correct and the answer is given to candidates on the List of formulae (MF19) provided. (3) Working is shown and the correct term is accepted for the answer, although the question asked for just the coefficient.

Answers are by real candidates in exam conditions. These show you the types of answers for each level. Discuss and analyse the answers with your learners in the classroom to improve their skills.

Examiner comments are alongside the answers. These explain where and why marks were awarded. This helps you to interpret the standard of Cambridge exams so you can help your learners to refine their exam technique.

Question 1

Example Candidate Response – 1	Examiner comments
1 Find the roots of the equation $z^3 = 7\sqrt{3} - 7i$, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi \leqslant \theta < \pi$. [5] $\cancel{z^3 = \cos(2k\pi - \frac{1}{6}\pi) + i\sin(2k\pi - \frac{1}{6}\pi)}$ $\cancel{z = \cos\left(\frac{2k\pi - \frac{1}{6}\pi}{3}\right) + i\sin\left(\frac{2k\pi - \frac{1}{6}\pi}{3}\right)}$ $z = 14e^{i\frac{2k\pi - \frac{1}{6}\pi}{3}}$ $z = 14e^{i\frac{2k\pi - \frac{1}{6}\pi}{3}}$ $k = 0, 1, 2$ $z = 14e^{i\frac{-1\pi}{18}}, 14e^{i\frac{11}{18}\pi}, 14e^{i\frac{23}{18}\pi}$ (1) (2) $z = 14e^{i\frac{-1}{18}\pi}, 14e^{i\frac{11}{18}\pi}, 14e^{i\frac{23}{18}\pi}$	(1) The argument given for the third root is outside the range specified in the question. This highlights a common error candidates make when the question is not read carefully. (2) The first two roots have the correct argument, but the modulus of z should be the cube root of 14. This common error can be avoided by showing clear working, starting from z^3 in exponential form with the correct argument and modulus. **Total mark awarded = 3 out of 5**

Question 2

Example Candidate Response – 1	Examiner comments
2 (a) Find the coefficient of x^2 in the Maclaurin's series for $-\ln\cos x$.	
~~f(x)~~: $f(0) = 0$ $\quad f'(x) = -\frac{1}{\cos x}\cdot -\sin x$	
$= \tan x$ **1**	**1** The first derivative seen here is correct and most candidates could derive this using their prior knowledge from Cambridge International AS & A Level Mathematics 9709.
$f'(0) = 0$ $\quad f''(x) = \sec^2 x$ **2**	**2** The second derivative is also correct and the answer is given to candidates on the List of formulae (MF19) provided.
$f''(0) = 1$	
$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0)$	
$= 0 + 0 + 1\cdot\frac{x^2}{2}$	
$= \frac{1}{2}x^2$ **3**	**3** Working is shown and the correct term is accepted for the answer, although the question asked for just the coefficient. Mark for (a) = 4 out of 4
(b) Find the length of the arc of the curve with equation $y = -\ln\cos x$ from the point where x the point where $x = \frac{1}{4}\pi$.	
$S = \int_0^{\frac{1}{4}\pi} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$	
$\frac{dy}{dx} = -\frac{1}{\cos x}\cdot(-\sin x)$ $\quad S = \int_0^{\frac{1}{4}\pi} \sqrt{1 + \tan^2 x}\, dx$	
$= \tan x$	
$S = \int_0^{\frac{1}{4}\pi} \sqrt{\sec^2 x}\, dx$	
$S = \int_0^{\frac{1}{4}\pi} \sec x\, dx$ **4**	**4** Good working is shown above, and the correct integral is formed using the formula for arc length; this is not provided on the List of formulae and candidates are required to memorise it.
$= \left[\tan^2 x\right]_0^{\frac{1}{4}\pi}$ **5**	**5** At this stage, it was common to see the integral of sec written incorrectly, as shown here, or the answer written using a calculator with no working. Note that the integral is provided to candidates on the List of formulae (MF19). Mark for (b) = 2 out of 4
$= 1$	
	Total mark awarded = 6 out of 8

Question 3

Example Candidate Response – 1	Examiner comments
3 The matrix **A** is given by $$\mathbf{A}=\begin{pmatrix}6 & -9 & 5\\ 5 & -8 & 5\\ 1 & -1 & 2\end{pmatrix}.$$	
(a) Find the eigenvalues of **A**.	
$(6-\lambda)(-8-\lambda)(2-\lambda)+5\times5\times(-1)+(-9)\times5\times1$ $-1\times(-8-\lambda)\times5-5\times(-9)\times(2-\lambda)-(6-\lambda)\times(-1)\times5$	
$=(-48+2\lambda+\lambda^2)(2-\lambda)-25-45+40+5\lambda+90-45\lambda$ $+30-5\lambda$	
$=-\lambda^3+52\lambda-96-45\lambda+90$. 1	1 This part of the question was answered well with the candidate showing full working when expanding and factorising. Mark for (a) = 4 out of 4
$=-\lambda^3+7\lambda-6=0$.	
$\lambda_1=1$, $(\lambda-1)(\lambda+3)(-\lambda-2)=0$.	
$\lambda_2=-3$,	
$\lambda_3=2$	
(b) Use the characteristic equation of **A** to show that $\mathbf{A}^{-1}=p\mathbf{A}^2+q\mathbf{I}$, where p and q are constants to be determined. [3]	
since $-\lambda^3+7\lambda-6=0$.	
$-A^3+7A-6I=0$	
$-A^2+7I-6A^{-1}=0$.	
$A^{-1}=\frac{1}{6}(-A^2+7I)$. 2	2 The candidate maintains accuracy and uses correct notation when manipulating the characteristic equation.
$A^2=\begin{pmatrix}6 & -9 & 5\\ 5 & -8 & 5\\ 1 & -1 & 2\end{pmatrix}\begin{pmatrix}6 & -9 & 5\\ 5 & -8 & 5\\ 1 & -1 & 2\end{pmatrix}=\begin{pmatrix}-4 & 13 & -5\\ -5 & 14 & -5\\ 3 & -3 & 4\end{pmatrix}$.	3 The candidate spends time substituting in for the matrices, which was not required to answer the question. This again highlights the importance of reading the question carefully and leaving the answer in the required form. Mark for (b) = 2 out of 3
$A^{-1}=\frac{1}{6}\left(\begin{pmatrix}4 & -13 & 5\\ 5 & -14 & 5\\ -3 & 3 & -4\end{pmatrix}+7\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right)$	
$=\frac{1}{6}\begin{pmatrix}11 & -13 & 5\\ 5 & -7 & 5\\ -3 & 3 & 3\end{pmatrix}$ 3	**Total mark awarded = 6 out of 7**

Question 4

Example Candidate Response – 1

4 It is given that

$$x = -t + \tan^{-1} t \qquad \text{and} \qquad y = t + \sinh^{-1} t.$$

(a) Show that $\dfrac{dy}{dx} = -\dfrac{t^2 + 1 + \sqrt{t^2+1}}{t^2}$.

$\cancel{\dfrac{dx}{dt} = -1 + \dfrac{1}{1+x^2} = \dfrac{-1-x^2+1}{1+x^2} = \dfrac{-x^2}{1+x^2}}$ $\quad \cancel{\dfrac{dy}{dt} = 1 + \dfrac{1}{\sqrt{1+x^2}} = \dfrac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}}}$

$\cancel{\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = \dfrac{\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}}}{\frac{-x^2}{1+x^2}} = \dfrac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}} \times \dfrac{1+x^2}{-x^2} = -\dfrac{1+x^2+\sqrt{1+x^2}}{x^2}}$

$\dfrac{dx}{dt} = -1 + \dfrac{1}{1+t^2}$ $\qquad \dfrac{dy}{dt} = 1 + \dfrac{1}{\sqrt{1+t^2}}$

$= \dfrac{-1-t^2+1}{1+t^2}$ $\qquad = \dfrac{\sqrt{1+t^2}+1}{\sqrt{1+t^2}}$

$= \dfrac{-t^2}{1+t^2}$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt}$

$= \dfrac{\frac{\sqrt{1+t^2}+1}{\sqrt{1+t^2}}}{\frac{-t^2}{1+t^2}}$

$= \dfrac{\sqrt{1+t^2}+1}{\sqrt{1+t^2}} \times \dfrac{1+t^2}{-t^2}$ 1

$= -\dfrac{1+t^2+\sqrt{1+t^2}}{t^2}$ (shown)

(b) Find the value of $\dfrac{d^2y}{dx^2}$ when $t = \frac{3}{4}$.

$\dfrac{dy}{dt} = -\dfrac{t^2+1+\sqrt{t^2+1}}{t^2}$

$= -\left(1 + \dfrac{1}{t^2} + \dfrac{\sqrt{t^2+1}}{t^2}\right)$

$= -1 - \dfrac{1}{t^2} + \dfrac{\sqrt{t^2+1}}{t^2}$

$= -1 - t^{-2} + \dfrac{(t^2+1)^{\frac{1}{2}}}{t^2}$

$\dfrac{d'y}{dt'} = 2t^{-3} + \dfrac{t^2 \cdot \frac{1}{2}(t^2+1)^{-\frac{1}{2}} \cdot 2t - 2t(t^2+1)^{\frac{1}{2}}}{t^4}$

$= 2t^{-3} + \dfrac{t^3(t^2+1)^{-\frac{1}{2}} - 2t(t^2+1)^{\frac{1}{2}}}{t^4}$ 2

$\dfrac{d^2y}{dx^2} = \dfrac{2t^{-3} + \frac{t^3(t^2+1)^{-\frac{1}{2}} - 2t(t^2+1)^{\frac{1}{2}}}{t^4}}{\left(\frac{-t^2}{1+t^2}\right)^2}$ 3

when $t = \frac{3}{4}$ $\quad \dfrac{d^2y}{dx^2} = -0.914$

Examiner comments

1 Most candidates used parametric differentiation correctly. However, it is required to show enough working to justify the given answer and the response here highlights the point where many candidates stopped prematurely.
Mark for (a) = 3 out of 4

2 The answer from part (a) is correctly differentiated with respect to t using prior knowledge from Cambridge International AS & A Level Mathematics 9709.

3 As for many candidates, applying the chain rule to find the second derivative is problematic and, here, we see a misconception that the first derivative of x with respect to t needs to be squared.
Mark for (b) = 2 out of 5

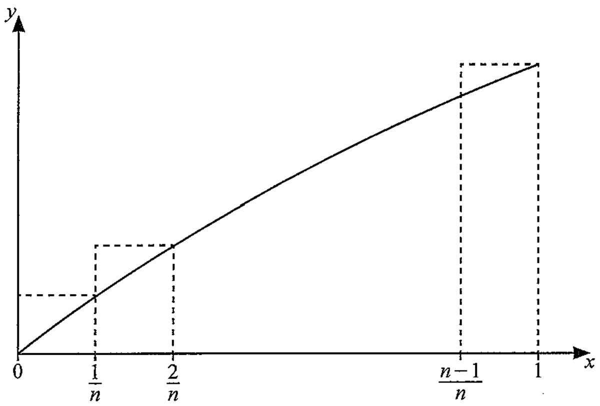
Total mark awarded = 5 out of 9

Question 5

Example Candidate Response – 1	Examiner comments
5 Find the solution of the differential equation $$x(x+7)\frac{dy}{dx}+7y=x$$ for which $y=7$ when $x=1$. Give your answer in the form $y=f(x)$. [9]	
$\boxed{\frac{dy}{dx}+F(x)y=G(x)}$ $x(x+7)\frac{dy}{dx}+7y=x \rightarrow \frac{dy}{dx}+\frac{7}{x(x+7)}y=\frac{x}{x(x+7)}$ $\frac{dy}{dx}+\frac{7}{x(x+7)}y=\frac{1}{(x+7)}$ $I(x)=e^{\int\frac{7}{x(x+7)}dx}=e^{7\int\frac{1}{x(x+7)}dx}$ $\frac{1}{x(x+7)}=\frac{A}{x}+\frac{B}{x+7}$ $7\int\frac{1}{x(x+7)}=7\int\frac{1}{7x}-\frac{1}{7(x+7)}dx$ $1=Ax+7A+Bx$ $=\int\frac{1}{x}-\frac{1}{(x+7)}dx$ $A+B=0$ $7A=1$ $=\ln x-\ln(x+7)=\ln\left(\frac{x}{x+7}\right)$ $A=-B$ $A=\frac{1}{7}$ $B=-\frac{1}{7}$ $I(x)=e^{\ln\left(\frac{x}{x+7}\right)}=\frac{x}{x+7}$ (1)	(1) As seen here, most candidates divided both sides of the equation by $\frac{x}{(x+7)}$ and found the integrating factor correctly by using partial fractions or completing the square.
$\frac{d}{dx}\left[\frac{x}{x+7}(y)\right]=\frac{1}{(x+7)}\times\frac{x}{(x+7)}$ $\frac{d}{dx}\left[\frac{x}{x+7}(y)\right]=\frac{x}{(x+7)(x+7)}$ $\frac{x}{x+7}(y)=\int\frac{x}{(x+7)(x+7)}dx=\int\frac{x}{(x+7)^2}dx$ $\rightarrow$ let $u=x$ $\frac{dv}{dx}=(x+7)^{-2}$ $\frac{du}{dx}=1$ $v=\frac{(x+7)^{-1}}{(1)(-2)}=-\frac{1}{2}(x+7)^{-1}$ (2) $=x\left(-\frac{1}{2}(x+7)^{-1}\right)-\int-\frac{1}{2}(x+7)^{-1}(1)\,dx$ $=-\frac{x}{2(x+7)}+\frac{1}{2}\int\frac{1}{x+7}dx=-\frac{x}{2(x+7)}+\frac{1}{2}\ln(x+7)+C$	(2) Many candidates made an error when applying integration by parts to the right-hand side. The incorrect multiplication of $\frac{1}{2}$ when finding v here emphasises the need to check all working thoroughly, particularly for high-tariff questions such as this.
$\frac{x}{x+7}(y)=-\frac{x}{2(x+7)}+\frac{1}{2}\ln(x+7)+C$ @ $y=7$, $x=1$ $\frac{1}{1+7}(7)=-\frac{1}{2(1+7)}+\frac{1}{2}\ln(1+7)+C$ $\frac{7}{8}=-\frac{1}{16}+\frac{1}{2}\ln 8$ $\frac{15}{16}=\frac{1}{2}\ln 8+C$ $C=\frac{15}{16}-\frac{1}{2}\ln 8$ $\frac{x}{x+7}(y)=-\frac{x}{2(x+7)}+\frac{1}{2}\ln(x+7)+\frac{15}{16}-\frac{1}{2}\ln 8$ $y=\frac{x+7}{x}\left[-\frac{x}{2(x+7)}+\frac{1}{2}\ln\left(\frac{x+7}{8}\right)+\frac{15}{16}\right]$ (3) $=-\frac{1}{2}+\frac{x+7}{x}\left[\frac{1}{2}\ln\left(\frac{x+7}{8}\right)+\frac{15}{16}\right]$	(3) Most candidates gained method marks for substituting the initial conditions and making y the subject. However, as seen here, the later accuracy marks were often lost due to errors, such as the one described above, from earlier working. **Total mark awarded = 6 out of 9**

Question 6

Example Candidate Response – 1	Examiner comments



The diagram shows the curve with equation $y = \ln(1+x)$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles each of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 \ln(1+x)\,dx < U_n$, where

$$U_n = \frac{1}{n}\ln\frac{(2n)!}{n!} - \ln n. \qquad [4]$$

$U_n = \sum_{r=1}^{n} \ln(1+x) = \frac{1}{n}\left(\ln(1+\frac{1}{n}) + \ln(1+\frac{2}{n}) + \dots + \ln(1+\frac{n}{n})\right)$

$U_n - \ln n = \frac{1}{n}\left(\ln(\frac{n+1}{n}) + \ln(\frac{n+2}{n}) + \dots + \ln(\frac{2n}{n})\right)$ 1

$U_n = \frac{1}{n}\ln\frac{(2n)!}{n!} - \ln n$ qed 2

(b) Use a similar method to find, in terms of n, a lower bound L_n for $\int_0^1 \ln(1+x)\,dx$.

$L_n = \frac{1}{n}\left(\ln(1+\frac{1}{n}) + \ln(1+\frac{2}{n}) + \dots + \ln(1+\frac{n-1}{n})\right)$ 3

$L_n = \frac{1}{n}\ln\frac{(2n)!}{n!} - \ln n - \ln(1+\frac{n}{n})$

$= \frac{1}{n}\ln\frac{(2n)!}{n!} - \ln\frac{1}{2}n$ qed 4

(c) By simplifying $U_n - L_n$, show that $\lim_{n\to\infty}(U_n - L_n) = 0$.

$\frac{1}{n}\ln\frac{(2n)!}{n!} - \ln n - \left(\frac{1}{n}\ln\frac{(2n)!}{n!} - \ln\frac{1}{2}n\right) = \ln\frac{1}{2}n - \ln n$

5

Examiner comments

1 A correct expression is formed here for the sum of the areas of the rectangles and enough terms, including the last, are written down.

2 Since the answer is given, application of the laws of logarithms needs to be explicit. As seen here, some candidates did not show enough working to gain the last two marks for this part.
Mark for (a) = 2 out of 4

3 The correct rectangles are used and enough terms, including the last, are written down.

4 The last term here has not been multiplied by $\frac{1}{n}$ and the logarithms have been incorrectly combined on the last line. Mistakes such as these were common and this again emphasises the need for candidates to check their work carefully.
Mark for (b) = 2 out of 4

5 Errors in the previous part often made it impossible for candidates to show that the difference is proportional to $\frac{1}{n}$, and hence justify the given limit. An answer such as the one seen here is a clear indication that the answer to part (b) is wrong.
Mark for (c) = 0 out of 2

Total mark awarded = 4 out of 10

Question 7

Example Candidate Response – 1	Examiner comments
7 The variables x and y are related by the differential equation $$4\frac{d^2y}{dx^2} - y = 3.$$ It is given that, when $x = 0$, $y = -3$ and $\frac{dy}{dx} = 2$. **(a)** Find y in terms of x. $4\lambda^2 - 1 = 0$ $\lambda = \pm\frac{1}{2}$ CF: $y = Ae^{\frac{1}{2}x} + Be^{-\frac{1}{2}x}$ PI: $y = \cancel{ax}\lambda$ $\frac{dy}{dx} = \cancel{0}\ \lambda$ $\frac{d^2y}{dx^2} = 0$ GS = CF + PI = $y = Ae^{\frac{1}{2}x} + Be^{-\frac{1}{2}x}$ when $y = -3$, $x = 0$ $-3 = A + B$, $\cancel{A = -3 - B}$ $\frac{dy}{dx} = \frac{1}{2}Ae^{\frac{1}{2}x} - \frac{1}{2}Be^{-\frac{1}{2}x}$ when $x = 0$, $\frac{dy}{dx} = 2$ $2 = \frac{1}{2}A - \frac{1}{2}B$ $A - B = 1$, $\cancel{A = 1 + B}$ $\cancel{3 = B}$ $\cancel{A + B =}$ $A + B = -3$, $B = -3 - A$ $A - B = 1$ $A + 3 + A = 1$ $\cancel{2A = -2}$ $A = -1$ $B = -2$ $y = -e^{\frac{1}{2}x} - 2e^{-\frac{1}{2}x}$	**1** As seen here, most candidates took the most efficient approach of substituting $y=c$ to find the particular integral. **2** There was some inaccuracy when solving linear equations to find the values of the constants and some problems with notation. Where errors occurred, such as in this response, it was quite common to see the particular integral set to zero, giving an incorrect solution. Mark for (a) = 4 out of 8
(b) Deduce the exact value of x for which $y = 0$. Give your answer in logarithmic form. $-e^{\frac{1}{2}x} - 2e^{-\frac{1}{2}x} = 0$ $e^{\frac{1}{2}x} = -2e^{-\frac{1}{2}x}$ $e^{\frac{1}{2}x} = e^{-\frac{1}{2}x + \ln(-2)}$ $\cancel{\ln(-2) =}$ $\ln(-2) - \frac{1}{2}x = \frac{1}{2}x$ $\ln(-2) = x$	**3** Those who were unsuccessful with part (a) were usually unable to derive and solve an equation of the form $\sinh(ax)=b$, which was the required approach for this part. The response shown here emphasises the importance of sense-checking answers. When arriving at an impossible answer, such as the one shown here, candidates should check their work for the whole question. Mark for (b) = 0 out of 3 **Total mark awarded = 4 out of 11**

Question 8

Example Candidate Response – 1

8 **(a)** Find $\int \sin\theta\cos^n\theta\,d\theta$, where $n \neq -1$. [2]

$\int \sin\theta\cos^n\theta\,d\theta$

$= -\frac{1}{n+1}\cos^{n+1}\theta + C$

$\frac{d}{dx}\cos^{n+1}\theta$

$= (n+1)\cos^n\theta(-\sin\theta)$

$= -(n+1)\sin\theta\cos^n\theta$

$\frac{d}{dx}(\cos^{n+1}\theta)$

$= -(n+1)\sin\theta\cos^n\theta$

1

Let $I_{m,n} = \int_0^{\frac{1}{2}\pi} \sin^m\theta\cos^n\theta\,d\theta$.

(b) Show that, for $m \geqslant 2$ and $n \geqslant 0$,

$$I_{m,n} = \frac{m-1}{m+n}I_{m-2,n}.$$ [5]

$\rightarrow \int \sin^{m-2}\theta\cos^n\theta$

$\int \cos^3\theta = \frac{1}{3}\cos^3\theta\sin\theta$

$I_{m,n} = \int_0^{\pi/2} \sin^m\theta\cos^n\theta\,d\theta$

$= \int_0^{\pi/2} \sin^{m-1}\theta\sin\theta\cos^n\theta\,d\theta$

$= -\frac{1}{n+1}\cos^{n+1}\theta\sin^{m-1}\theta + \int \frac{m-1}{n+1}\sin^{m-2}\theta\cos^{n+1}\theta$

$u = \sin^{m-1}\theta \quad \frac{dv}{d\theta} = \sin\theta\cos^n\theta$

$\frac{du}{d\theta} = (m-1)\sin^{m-2}\theta \quad v = -\frac{1}{n+1}\cos^{n+1}\theta$

$\cos\theta$

$= -\frac{1}{n+1}\cos^{n+1}\sin^{m-1}\theta + \int \frac{m-1}{n+1}\sin^{m-2}\theta\cos^{n+1}\theta\cos\theta\,d\theta$

$=$ 2

~~$= \int_0^{\frac{\pi}{2}} \sin^{m-2}\sin^2\theta\cos^n\theta\,d\theta = \int_0^{\frac{\pi}{2}} \sin^{m-1}\theta\sin\theta\cos\theta\cos^{n-1}\theta\,d\theta$~~

~~$= \int_0^{\frac{\pi}{2}} \sin^m\theta\cos\theta\cos^{n-1}\theta\,d\theta$~~ $= \int_0^{\frac{\pi}{2}} \sin^{m-1}\theta\sin\theta\cos^{n-1}\theta\cos\theta\,d\theta$

(c) By considering the binomial expansion of $\left(z+\frac{1}{z}\right)^5$, where $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$\cos^5\theta = a\cos 5\theta + b\cos 3\theta + c\cos\theta,$$

where a, b and c are constants to be determined. [5]

~~$(z+\frac{1}{z})^5 = (\cos 2\theta)^5 = \cos(2\cos^2\theta - 1)^5$~~

$(2\cos\theta)^5 = (z+\frac{1}{z})^5 \rightarrow z+\frac{1}{z} = 2\cos\theta$

$2^5\cos^5\theta = z^5 + 5z^4(\frac{1}{z}) + 10z^3(\frac{1}{z})^2 + 10z^2(\frac{1}{z})^3 + 5z(\frac{1}{z})^4 + (\frac{1}{z})^5$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$32\cos^5\theta = z^5 + \frac{1}{z^5} + 5z^3 + 5z^{-3} + 10z + 10z^{-1}$

$32\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$

$\cos^5\theta = \frac{1}{32}\cos 5\theta + \frac{5}{32}\cos 3\theta + \frac{5}{16}\cos\theta$ 3

$\therefore a = \frac{1}{32},\ b = \frac{5}{32},\ c = \frac{5}{16}$

(d) Using the results given in parts **(b)** and **(c)**, find the exact value of $I_{2,5}$.

$I_{2,5} = \int_0^{\pi/2} \sin^2\theta\cos^5\theta\,d\theta = \frac{2-1}{2+5} I_{0,5} = \frac{1}{7}\int_0^{\pi/2} \cos^5\theta\,d\theta$

$I_{2,5} = \frac{1}{7}\int_0^{\pi/2} \cos^5\theta\,d\theta$

$= \frac{1}{7}\int_0^{\pi/2} \frac{1}{32}\cos 5\theta + \frac{5}{32}\cos 3\theta + \frac{5}{16}\cos\theta\,d\theta$

$= \frac{1}{7}\left[\frac{1}{160}\cancel{\cos}\sin 5\theta + \frac{5}{96}\sin 3\theta + \frac{5}{16}\sin\theta\right]_0^{\pi/2}$ 4

$= \frac{1}{7}\left[\left(\frac{1}{160} - \frac{5}{96} + \frac{5}{16}\right) - (0)\right]$

$= \frac{1}{7}\left(\frac{4}{15}\right)$

$= \frac{4}{105}$

Examiner comments

1 Almost all candidates recognised the integral, although it was common to see the arbitrary constant omitted.
Mark for (a) = 2 out of 2

2 As seen here, most candidates integrated by parts first, using the result from part (a) to inform their choice of parts. To progress further, it is required to apply a standard trigonometric identity provided to candidates on the List of formulae (MF19).
Mark for (b) = 1 out of 5

3 After expanding using the binomial expansion, terms are grouped together clearly. As in this response, there were often errors in the denominators of the determined constants which again emphasises the importance of checking working thoroughly.
Mark for (c) = 3 out of 5

4 The candidate applies the reduction formula from part (b) and the result from part (c). As seen here, inaccuracy from part (c) led to an incorrect answer here.
Mark for (d) = 2 out of 4

Total mark awarded = 8 out of 16

Cambridge Assessment International Education
The Triangle Building, Shaftesbury Road, Cambridge, CB2 8EA, United Kingdom
t: +44 1223 553554
e: info@cambridgeinternational.org   www.cambridgeinternational.org