Cambridge Assessment International Education

Example Candidate Responses – Paper 1

Cambridge International AS & A Level Further Mathematics 9231

For examination from 2022

Cambridge Pathway

Contents

Introduction

The main aim of this booklet is to exemplify standards for those teaching Cambridge International AS & A Level Further Mathematics 9231 and to show how different levels of candidates' performance relate to the subject's curriculum and assessment objectives.

In this booklet, candidate responses have been chosen from the June 2022 series to exemplify a range of answers for all the questions on the question paper.

For each question, the response is annotated with a clear explanation of where and why marks were awarded or omitted. In this way, it is possible for you to understand what candidates have done to gain their marks and what they could do to improve their answers.

This document provides illustrative examples of candidate work with examiner commentary. These help teachers to assess the standard required to achieve marks beyond the guidance of the mark scheme. Please also refer to the June 2022 Examiner Report for further detail and guidance.

The questions and mark schemes used here are available to download from the School Support Hub. These files are:

9231 June 2022 Question Paper 13

9231 June 2022 Mark Scheme 13

Past exam resources and other teaching and learning resources are available on the School Support Hub:

www.cambridgeinternational.org/support

How to use this booklet

This booklet goes through the paper one question at a time. The candidate answers are set in a table. In the left-hand column are the candidate answers, and in the right-hand column are the examiner comments.

Example Candidate Response – 1, continued	Examiner comments
Let $\mathbf{B} = \begin{pmatrix} b & b \\ a^{-1} & a^{-1} \end{pmatrix}$, where b is a positive constant. **(c)** Find the equations of the invariant lines, through the origin, of the transformation in the x–y plane represented by $\mathbf{A}^n\mathbf{B}$. [6] $A^nB = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} b & b \\ a^{-1} & a^{-1} \end{pmatrix} = \begin{pmatrix} b+n & b+n \\ a^{-1} & a^{-1} \end{pmatrix}$, **7** $\therefore \begin{pmatrix} b+n & b+n \\ a^{-1} & a^{-1} \end{pmatrix} \begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} T \\ mT \end{pmatrix}$. **8** **9** $b+n+(b+n)m$	**7** Correct matrices and result. **8** This shows the candidate is looking for invariant line and not invariant point. **9** The candidate transforms the point, and a correct equation is

Answers are by real candidates in exam conditions. These show you the types of answers for each level. Discuss and analyse the answers with your learners in the classroom to improve their skills.

Examiner comments are alongside the answers. These explain where and why marks were awarded. This helps you to interpret the standard of Cambridge exams so you can help your learners to refine their exam technique.

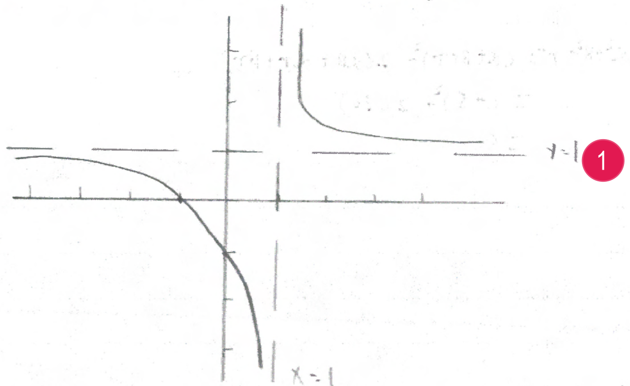
Question 1

Example Candidate Response – 1

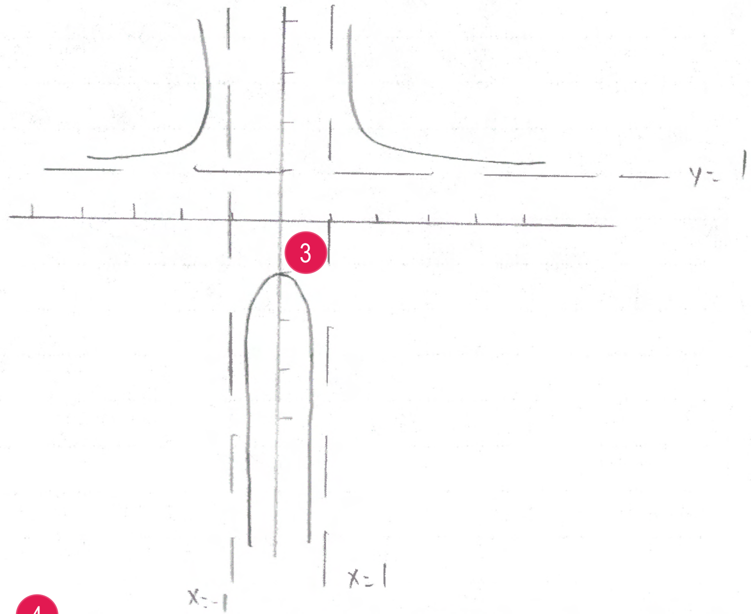
1 (a) Sketch the curve with equation $y = \frac{x+1}{x-1}$. [2]

$(0,-1)$ $(-1,0)$

$x=1$ $y=1$



(b) Sketch the curve with equation $y = \frac{|x|+1}{|x|-1}$ and find the set of values of x for which $\frac{|x|+1}{|x|-1} < -2$. [4]



4 $-2 = \frac{x+1}{x-1}$ $-2x+2 = x+1$ $3x=1$ $x=\frac{1}{3}$ $\frac{1}{3} < x < 1$ 5

$-\frac{1}{3} < x < -1$ 6

Examiner comments

1 The asymptotes are correct and clearly identified. Candidates should ensure they label lines and significant points on every graph.

2 Every time they draw a graph approaching an asymptote, candidates should be careful that the curve approaches the line steadily and does not appear to cross it. Here the curve is getting further from the asymptote.
Mark for (a) = 2 out of 2

3 Candidates need to check the shape of a curve when it meets the line in which it is reflected. In this case (0, –1) must be shown as a cusp not a turning value. It is often easier to draw if one side is drawn using the original graph and then the second side is drawn separately.

4 Here they are using the positive value of x and an equation to find the critical value.

5 This is the correct inequality for positive values of x. Some candidates add the line $y = -2$ to the graph to make it easier to read off the inequalities.

6 Mistakes with inequalities are common. Here they are using the symmetry of the graph, but should check that both inequalities make sense.
Mark for (b) = 2 out of 4

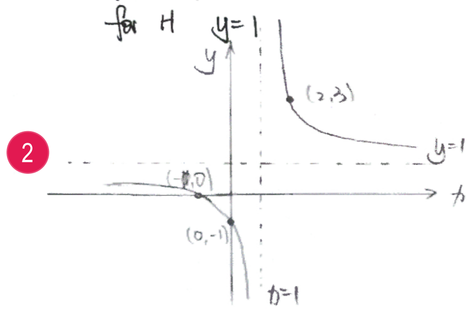
Total mark awarded = 4 out of 6

Example Candidate Response – 2

1 (a) Sketch the curve with equation $y = \frac{x+1}{x-1}$. [2]

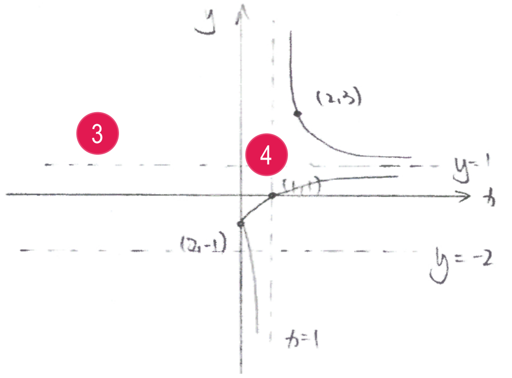
for V $x=1$
for H $y=1$

when $x=0$, $y=-1$
when $x=2$, $y=3$
when $x=-1$, $y=0$



(b) Sketch the curve with equation $y = \frac{|x|+1}{|x|-1}$ and find the set of values of x for which $\frac{|x|+1}{|x|-1} < -2$. [4]

for $y = \frac{|x|+1}{|x|-1}$



for $\frac{|x|+1}{|x|-1} < -2$ $\quad y < -2 \quad 3x < 1$ (6)

(5) $x+1 < -2\cdot(x-1) \quad x < \frac{1}{3}$

$x+1 < -2x+2$

$3x-1 < 0$

Examiner comments

1 The asymptotes are correct and clearly labelled. Candidates should ensure they label lines and significant points on every graph.

2 Every time they draw a graph approaching an asymptote, candidates should be careful that the curve approaches the line steadily and does not appear to cross it. Here, they need to show a closer approach to the asymptote.
Mark for (a) = 2 out of 2

3 Candidates should check that the graph applies for all values of x. Here, they need to show what happens when x takes negative values.

4 This part of the graph should not appear. A check of one point lying on it, for example (1, 0), shows that it does not satisfy the equation.

5 The candidate considers the positive value of x. For the inequality to remain valid when multiplied by $(x - 1)$, $(x - 1)$ must be positive. It is often easier to work with an equation rather than an inequality to find the critical values.

6 This is the correct critical value for x but the wrong sign. The graph should show which inequality is needed.
Mark for (b) = 1 out of 4

Total mark awarded = 3 out of 6

Question 2

Example Candidate Response – 1

The cubic equation $x^3 + 5x^2 + 10x - 2 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$.

$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2(\Sigma\alpha\beta)$

$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)$ 1

$= \left(-\frac{5}{1}\right)^2 - 2\left(\frac{10}{1}\right)$

$= 5$

(b) Show that the matrix $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$ is singular.

$ad - bc$

$\det = 1\begin{vmatrix} 1 & \gamma \\ \gamma & 1 \end{vmatrix} - \alpha\begin{vmatrix} \alpha & \gamma \\ \beta & 1 \end{vmatrix} + \beta\begin{vmatrix} \alpha & 1 \\ \beta & \gamma \end{vmatrix}$

$= 1(1 - \gamma^2) - \alpha(\alpha - \beta\gamma) + \beta(\alpha\gamma - \beta)$

$= 1 - \gamma^2 - \alpha^2 + \beta\gamma\alpha + \alpha\beta\gamma - \beta^2$

$= 1 - (\alpha^2 + \beta^2 + \gamma^2) + 2\alpha\beta\gamma$ 2

$= 1 - (5) + 2(-2)$ 3

$= 0$

Examiner comments

1 The candidate clearly shows the way of finding the sums from the coefficients.
Mark for (a) = 3 out of 3

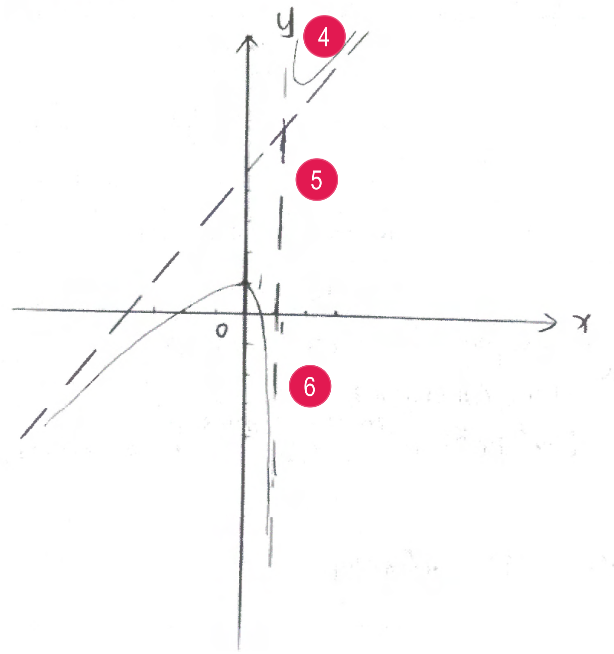
2 The candidate correctly evaluates the determinant with full working shown.

3 Here, the method mark includes correctly finding the product of the roots. The working shows that the candidate uses -2. They need to show that they are substituting +2 for this, and *their* answer for 2(a) to be awarded the mark.
Mark for (b) = 2 out of 4

Total mark awarded = 5 out of 7

Question 3

Example Candidate Response – 1	Examiner comments
3 A curve C has equation $y = \frac{ax^2 + x - 1}{x - 1}$, where a is a positive constant. (a) Find the equations of the asymptotes of C. $x - 1 = 0$ $x = 1$ $x-1 \overline{)\, ax^2 + x - 1}$ quotient $ax + 1 + a$ $ax^2 - ax$ $(1+a)x - 1$ $(1+a)x - 1 - a$ a $y = ax + 1 + a$ $x = 1$ (1)	(1) Both asymptotes are written as equations. Mark for (a) = 3 out of 3
(b) Show that there is no point on C for which $1 < y < 1 + 4a$. $yx - y = ax^2 + x - 1$ $ax^2 + (1-y)x - 1 + y = 0$ $\Delta = (1-y)^2 - 4a(-1+y) < 0$ (2) $1 - 2y + y^2 + 4a - 4ay < 0$ $y^2 + (-4a - 2)y + 4a + 1 < 0$ $1 < y < 1 + 4a$ (3)	(2) This is the most usual method. The candidate identifies when there are no possible y values by using the correct discriminant less than zero. (3) The candidate now needs to relate the inequality above to $1 < y < 1 + 4a$. They could either factorise the discriminant or draw a sketch graph to show that this inequality holds. Mark for (b) = 2 out of 4

Example Candidate Response – 1, continued	Examiner comments
(c) Sketch C. You do not need to find the coordinates of the intersections with the axes. 	4 Here, the curve approached the asymptote, but curved away again. Every time the candidate draws a graph approaching an asymptote, they should be careful that the curve approaches the line steadily and does not appear to cross it. 5 Asymptotes look correct, but must be identified. The candidate could write on the equations or show the intersections with the axes. 6 Both shape and position of this branch are good. Mark for (c) = 1 out of 3 **Total mark awarded = 6 out of 10**

Example Candidate Response – 2	Examiner comments
6 3 A curve C has equation $y = \frac{ax^2+x-1}{x-1}$, where a is a positive constant. (a) Find the equations of the asymptotes of C. $x=1$ $x-1 \overline{)ax^2+x-1}$ quotient $ax+(a+1)$ ax^2-ax $\cancel{x+ax}-1$ $(a+1)x$ $(a+1)x-(a+1)$ $\cancel{1+a+1}\,a$. So $y=ax+a+1$. $x=1$	1 Both asymptotes are correctly shown as equations. Mark for (a) = 3 out of 3
(b) Show that there is no point on C for which $1 < y < 1+4a$. ~~as $y=0$~~ ~~so $ax^2+x-1=0$~~ ~~$\Delta = 1-4\times a\times(-1) = 1+4a$~~ ~~so $x=\frac{-1\pm\sqrt{1+4a}}{2a}$~~ ~~as $x=0$, $y=\frac{-1}{-1}=1$~~ ~~so $y \leq 1$~~ as $\frac{dy}{dx}=(x-1)(2ax+1)-(ax^2+x-1)\cdot 1$ $=2ax^2+x-2ax-1-ax^2-x+1$.	2 The candidate finds maximum and minimum values of y. The denominator is missing in the quotient rule.

Example Candidate Response – 2, continued

$\frac{dy}{dt} = at^2 - 2at$

as $\frac{dy}{dt} = 0$

$t(t-2) = 0$

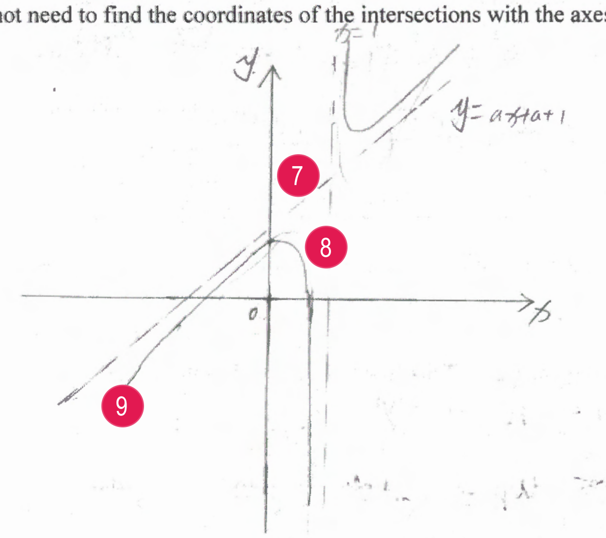
$t = 0$ or $t = 2$.

as $t = 2$, $y = \frac{4a+2-1}{2-1} = \frac{4a+1}{1} = 4a+1$

as $t = 0$, $y = \frac{-1}{-1} = 1$

so $1 < y < 1+4a$

(c) Sketch C. You do not need to find the coordinates of the intersections with the axes.



Examiner comments

3 These are the correct x values for the stationary points.

4 The candidate has the correct coordinates of stationary points and now needs to relate them to the possible values of y.

5 Now the candidate should show which is a maximum value and which is a minimum value.

6 To complete the solution by this method, the candidate needs to explain why there are no points on the graph between the maximum and minimum. They could say that the graph has two branches separated by an asymptote.
Mark for (b) = 1 out of 4

7 Correct asymptotes are clearly labelled.

8 The position and shape are good.

9 This is a common error. When drawing a curve with an asymptote, the curve should always make a consistent approach to asymptote. This one begins to get further away from the line.
Mark for (c) = 2 out of 3

Total mark awarded = 6 out of 10

Question 4

Example Candidate Response – 1	Examiner comments

4 Let $u_r = e^{rx}(e^{2x} - 2e^x + 1)$.

(a) Using the method of differences, or otherwise, find $\sum_{r=1}^{n} u_r$ in terms of n and x. [3]

$\cancel{u_r = e^{2+x^2} - 2e^{1+x^2} + e^{1+x}}$

$\cancel{\sum_{r=1}^{n} u_r = e^{2x^2} - 2e^{x^2} + e^{x} + e^{4x^2} - 2e^{2x^2} + e^{2x} + e^{6x^2} - 2e^{3x^2} + e^{3x}}$

$u_r = e^{(2+r)x} - 2e^{(r+1)x} + e^{rx}$

$\sum_{r=1}^{n} u_r = [e^{3x} - 2e^{2x} + e^{x}] + [e^{4x} - 2e^{3x} + e^{2x}] + [e^{5x} - 2e^{4x} + e^{3x}] + \ldots$ 1

$\cancel{+ e^{(1+n)x} - 2e^{(r+1)x} + e^{(r+2)x}} + [e^{(1+n)x} - 2e^{nx} + e^{(n-1)x}] + [e^{(2+n)x}$

$- 2e^{(n+1)x} + e^{nx}]$

$= e^{x} - e^{2x} - e^{(n+1)x} + e^{(2+n)x}$ 2

(b) Deduce the set of non-zero values of x for which the infinite series

$$u_1 + u_2 + u_3 + \ldots$$

is convergent and give the sum to infinity when this exists. [3]

when $n \to \infty$, $-e^{(n+1)x} \to -\infty$, 3

$e^{(n+2)x} \to \infty$.

So the infinite series is convergent.

$\sum_{r=1}^{n} u_r = e^{x} - e^{2x} - e^{(n+1)x} + e^{(n+1)x} \cdot e^{x}$

$= 0$ 4

(c) Using a standard result from the list of formulae (MF19), find $\sum_{r=1}^{n} \ln u_r$ in terms of n and x. [3]

$\sum_{r=1}^{n} \ln u_r$ 5

$= \ln e^{x} - \ln e^{2x} - \ln e^{(n+1)x} + \ln e^{(2+n)x}$ 6

$= x - 2x - (n+1)x + (n+2)x$

$= 0$

Examiner comments

1 The first three terms are shown correctly, and the last two terms are correct. The candidate needs to show carefully how the terms cancel out. As there are three terms for each value of r, a group of three terms must be shown, as well as the complete last term. It is often easier to see the pattern when a separate line is used for each value of r.

2 The candidate simplifies this to the right answer. This does not need to be simplified further.
Mark for (a) = 3 out of 3

3 For convergence of the infinite series, both terms involving n must tend to zero. The candidate is looking for values of x that make this happen.

4 Here, the candidate seems to be saying that the difference of two infinite quantities is zero.
Mark for (b) = 0 out of 3

5 It is better to use u_r in the form it is given in the question. Then, there is a multiplication to be changed to addition by taking natural logs.

6 It was a common error to see the logarithm of a sum as the sum of the logarithms.
Mark for (c) = 0 out of 3

Total mark awarded = 3 out of 9

Example Candidate Response – 2

4 Let $u_r = e^{rx}(e^{2x} - 2e^{x} + 1)$.

(a) Using the method of differences, or otherwise, find $\sum_{r=1}^{n} u_r$ in terms of n and x. [3]

$\sum_{r=1}^{n} u_r = (e^{3x} - 2e^{2x} + e^{x}) + (e^{4x} - 2e^{3x} + e^{2x}) + \dots + (e^{(2+n)x} - 2e^{(n+1)x} + e^{nx})$ 1

$= e^{x} - e^{2x} - e^{(n+1)x} + e^{(n+2)x}$ 2

$= e^{x}(1 - e^{x} - e^{nx} + e^{(n+1)x})$ 3

(b) Deduce the set of non-zero values of x for which the infinite series

$$u_1 + u_2 + u_3 + \dots$$

is convergent and give the sum to infinity when this exists. [3]

$\sum_{r=1}^{n} u_r$, for $n \to \infty$.

$S_\infty = e^{x} - e^{2x} - e^{(n+1)x} + e^{(n+2)x}$ $n \to \infty$. 4

~~for this to be convergent, $e^{(n+2)x} - e^{(n+1)x}$~~

~~$S_\infty = e^{x} - e^{2x} + e^{(n+1)x}(-1 + e^{x})$~~

~~$e^{x} - 1 = 0$ $e^{x} = 1$ $x = \ln 1 = 0$~~

~~$e^{(n+2)x} - e^{(n+1)x} = e^{nx} \cdot e^{2x} - e^{nx} \cdot e^{x}$~~

~~$= e^{nx}(e^{2x} - e^{x})$~~

~~for $n \to \infty$, $e^{nx} \to \infty$ e^{2x} e^{x} $= 0$~~

when $n \to \infty$, ~~$(n+1)$~~ $n+1$ ~~$\approx n$~~ $\approx n+2$. 5

So for $x \ll n$. $\Rightarrow x \ll \infty$.

S_∞ is convergent.

$S_\infty = e^{x} - e^{2x}$ 6

(c) Using a standard result from the list of formulae (MF19), find $\sum_{r=1}^{n} \ln u_r$ in terms of n and x. [3]

$\ln u_r = \ln e^{rx}(e^{2x} - 2e^{x} + 1)$ 7

$= \ln e^{rx} + \ln(e^{2x} - 2e^{x} + 1)$

$= rx + e^{x}(e^{x} - 2) - \frac{(e^{x}(e^{x}-2))^2}{2} + \frac{(e^{x}(e^{x}-2))^3}{3} + \dots$

8 9

Examiner comments

1 Each term involves three different powers of e. The candidate needs three consecutive values of r to show how the cancellation works. The last term is correct.

2 Many candidates find it easier to write the term for each value of r on a new line to show the pattern for cancellation.

3 The answer is correct. The method is incomplete, so 1 mark is awarded as a special case.
Mark for (a) = 1 out of 3

4 The candidate needs to show that both terms involving n tend to zero. They should look at values of x that make this happen.

5 The candidate tries to say the two infinities cancel out.

6 This is the correct answer, but it is not proven.
Mark for (b) = 0 out of 3

7 Original form for the u_r and correct use of the laws of logarithms.

8 The second term is the same for all values of r and so could be easily summed.

9 Simplification of the first term is correct, but this needs to be summed.
Mark for (c) = 1 out of 3

Total mark awarded = 2 out of 9

Question 5

Example Candidate Response – 1

5 Let $\mathbf{A} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, where a is a positive constant.

(a) State the type of the geometrical transformation in the x–y plane represented by $\mathbf{A}$.

Stretch (1)

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}.$$

Let P_k be the statement. $A^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}$ (2)

P_0: $n=1$ $A^1 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ true (3)

Assume that P_k is true $A^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}$ (4)

(5)

$\therefore P_{k+1}$: $A^{n+1} = A^n \times A = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (n+1)a \\ 0 & 1 \end{pmatrix}$ true,

Since P_0 is true, $P_k \Rightarrow P_{k+1}$, so P_k is true (6)

Examiner comments

1 The answer must be "shear".
Mark for (a) = 0 out of 1

2 The candidate should use consistent notation. k and n are both used here and throughout the proof.

3 Base case stated true with the matrix shown.

4 The candidate assumes the statement is true for k, with the matrix form shown.

5 The correct matrices are multiplied. The candidate needs to show the matrix with $a + na$ before writing down the answer they are trying to prove.

6 This final statement shows the implication. To complete the proof, the candidate should write the matrix form of the result and say it holds for all positive integers n.
Mark for (b) = 3 out of 5

Example Candidate Response – 1, continued

Let $\mathbf{B} = \begin{pmatrix} b & b \\ a^{-1} & a^{-1} \end{pmatrix}$, where b is a positive constant.

(c) Find the equations of the invariant lines, through the origin, of the transformation in the x–y plane represented by $\mathbf{A}^n\mathbf{B}$. [6]

$A^nB = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} b & b \\ a^{-1} & a^{-1} \end{pmatrix} = \begin{pmatrix} b+n & b+n \\ a^{-1} & a^{-1} \end{pmatrix}$. 7

$\therefore \begin{pmatrix} b+n & b+n \\ a^{-1} & a^{-1} \end{pmatrix} \begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} T \\ mT \end{pmatrix}$. 8

9 $\dfrac{b+n+(b+n)m}{a^{-1}+a^{-1}m} = \dfrac{1}{m}$.

$(b+n)m + (b+n)m^2 = a^{-1} + a^{-1}m$. 10

$(b+n)m^2 + (b+n-a^{-1})m - a^{-1} = 0$.

$\Delta = (b+n-a^{-1})^2 - 4(b+n)\cdot(-a^{-1})$

$= (b+n+a^{-1})^2$

$m = \dfrac{-(b+n-a^{-1}) \pm \sqrt{(b+n+a^{-1})^2}}{2(b+n)}$

$m = \dfrac{-b-n+a^{-1}-b-n-a^{-1}}{2(b+n)} = \dfrac{-2b-2n}{2b+2n} = -1$.

or $m = \dfrac{-b-n+a^{-1}+b+n+a^{-1}}{2(b+n)} = \dfrac{2a^{-1}}{2b+2n} = \dfrac{a^{-1}}{b+n}$

$\therefore y = -x$ or $y = \dfrac{a^{-1}}{b+n}x$ 11

Examiner comments

7 Correct matrices and result.

8 This shows the candidate is looking for invariant line and not invariant point.

9 The candidate transforms the point, and a correct equation is formed.

10 Correct equation and $(1 + m)$ is a factor on both sides. Factorising would avoid the hard work of using the quadratic formula.

11 The candidate provides accurate algebra giving both correct answers.
Mark for (c) = 6 out of 6

Total mark awarded = 9 out of 12

Example Candidate Response – 2

5 Let $\mathbf{A} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, where a is a positive constant.

(a) State the type of the geometrical transformation in the x–y plane represented by $\mathbf{A}$. [1]

shearing

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}.$$ [5]

Let P_k be the statement that, for some value $n=k$ (1)

P_1: $A^1 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = A \quad \therefore P_1$ is true (2)

$n=k$: $A^k = \begin{pmatrix} 1 & ka \\ 0 & 1 \end{pmatrix}$ (3)

$n=k+1$: $A \cdot A^k = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & ka \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & ka+a \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & a(k+1) \\ 0 & 1 \end{pmatrix} = A^{k+1}$ (4)

$\therefore P_k \Rightarrow P_{k+1}$ (5)

$\therefore$ by mathematical induction n is true for all positive integers (6)

Examiner comments

Mark for (a) = 1 out of 1

(1) This needs to make clear what the statement is.

(2) Case for n = 1 stated to be true with matrix shown.

(3) This needs to say, "assume that".

(4) The matrix multiplication is shown well.

(5) It is good to see the implication written down.

(6) This should show the result in matrix form.

Mark for (b) = 3 out of 5

Example Candidate Response – 2, continued

Let $\mathbf{B} = \begin{pmatrix} b & b \\ a^{-1} & a^{-1} \end{pmatrix}$, where b is a positive constant.

(c) Find the equations of the invariant lines, through the origin, of the transformation in the x–y plane represented by $\mathbf{A}^n\mathbf{B}$. [6]

$$A^nB = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b & b \\ a^{-1} & a^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} b + \frac{na}{a} & b + \frac{na}{a} \\ 0 + a^{-1} & 0 + a^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} b+n & b+n \\ a^{-1} & a^{-1} \end{pmatrix}$$ 7

$$\begin{pmatrix} b+n & b+n \\ a^{-1} & a^{-1} \end{pmatrix}\begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} T \\ mT \end{pmatrix}$$ 8

9 $\begin{cases} (b+n)t + (b+n)mt = T & ① \\ a^{-1}t + a^{-1}mt = mT & ② \end{cases}$

$\frac{①}{②}$ $\quad \frac{b+n+(b+n)m}{a^{-1} + a^{-1}m} = \frac{1}{m}$

$bm + nm + bm^2 + nm^2 = a^{-1} + a^{-1}m$ 10

$(b+n)m^2 + (b+n-a^{-1})m - a^{-1} = 0 \quad (m-a^{-1})(m+1)=0 \quad b+n \quad -a^{-1}$

$-a^{-1} + m = 0 \quad \begin{cases} m_1 = \cancel{\frac{b+n}{a}}\ a^{-1} \\ m_2 = -1 \end{cases} \quad 1 \quad 1$ 11

$m = a^{-1}$

$m + 1 = 0 \quad \begin{cases} y_1 = \cancel{(b+n)x} \\ y_2 = \cancel{x}\ -x \end{cases}$

$m = -1$

Examiner comments

7 The correct matrix multiplication is well explained.

8 This shows the candidate is looking for invariant lines, not invariant points.

9 The candidate transforms the point correctly.

10 It is always worth looking for a common factor. This candidate has noticed that the equation has $(1 + m)$ as a factor.

11 There is an error in factorisation.
Mark for (c) = 5 out of 6

Total mark awarded = 9 out of 12

Question 6

Example Candidate Response – 1	Examiner comments

6 The curve C has Cartesian equation $x^2 + xy + y^2 = a$, where a is a positive constant.

(a) Show that the polar equation of C is $r^2 = \frac{2a}{2 + \sin 2\theta}$. [3]

$x^2 + y^2 + xy = a.$

$\therefore r^2 + r\cos\theta \, r\sin\theta = a.$

$r^2 (1 + \cos\theta\sin\theta) = a.$

$r^2 (2 + 2\cos\theta\sin\theta) = 2a.$ 1

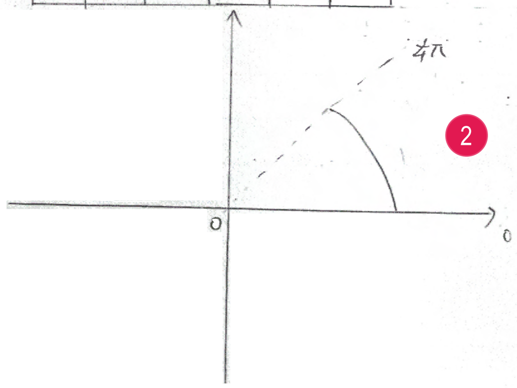
$r^2 (2 + \sin 2\theta) = 2a$

$r^2 = \frac{2a}{2 + \sin 2\theta}.$

1 This is a good and thorough proof. All cartesian terms have been changed to their polar forms. It is clear that the double angle formula has been used correctly.
Mark for (a) = 3 out of 3

(b) Sketch the part of C for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$. [2]

θ	0	$\frac{1}{12}\pi$	$\frac{1}{8}\pi$	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$
r	$\sqrt{a}$	$.9\sqrt{a}.$	$0.85\sqrt{a}$	$0.83\sqrt{a}$	$0.81\sqrt{a}$



2

2 This is all accurate. It is clearly a polar graph on the correct domain and of the right shape. The gradients at the extreme points are correct.
Mark for (b) = 2 out of 2

Example Candidate Response – 1, continued

The region R is enclosed by this part of C, the initial line and the half-line $\theta = \frac{1}{4}\pi$.

(c) It is given that $\sin 2\theta$ may be expressed as $\frac{2\tan\theta}{1+\tan^2\theta}$. Use this result to show that the area of R is

$$\frac{1}{2}a\int_0^{\frac{1}{4}\pi} \frac{1+\tan^2\theta}{1+\tan\theta+\tan^2\theta}\,d\theta$$

and use the substitution $t = \tan\theta$ to find the exact value of this area. [8]

$$S = \int_0^{\frac{1}{4}\pi} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{\frac{1}{4}\pi} \frac{1}{2} \cdot \frac{2a}{2+\sin 2\theta}\, d\theta$$

$$= \frac{1}{2}a\int_0^{\frac{1}{4}\pi} \frac{2}{2+\frac{2\tan\theta}{1+\tan^2\theta}}\, d\theta$$

3

$$= \frac{1}{2}a\int_0^{\frac{1}{4}\pi} \frac{1+\tan^2\theta}{1+\tan\theta^2+\tan\theta}\, d\theta$$

$t = \tan\theta$

$$\frac{dt}{d\theta} = \sec^2\theta = 1+\tan^2\theta = 1+t^2$$

$$= \frac{1}{2}a\int_0^{1} \frac{1+t^2}{1+t^2+t} \cdot \frac{1}{1+t^2}\, dt$$

4

$\cancel{\ln(1+t^2+t)}$

$\cancel{\frac{1}{1+t^2+t} \cdot 2t+1}$

$$= \frac{1}{2}a\int_0^{1} \frac{1}{1+t^2+t}\, dt$$

$$= \frac{1}{2}a\left[\frac{1}{2t+1}\ln(1+t^2+t)\right]_0^1$$

5

$$= \frac{1}{2}a\left(\frac{1}{2\times1+1}\ln(1+1+1)\right) - \frac{1}{2}a\left(\frac{1}{2\times0+1}\ln(1+0+0)\right)$$

$$= \frac{1}{6}a\ln 3 - 0$$

$$= \frac{1}{6}a\ln 3 = \frac{a}{6}\ln 3$$

Examiner comments

3 The candidate needs to show why this integral is equal to the given answer. They are proving a given result and should put in an intermediate step to move from the line with a fraction in the denominator, to the integral given in the question.

4 This is well done – the stages of the substitution are shown clearly, and the new limits are correct.

5 This was a very common error. Here, the candidate needs to complete the square to make the function into one of the standard integrals that are given in the List of formulae (MF19).
Mark for (c) = 3 out of 8

Total mark awarded = 8 out of 13

Example Candidate Response – 2	Examiner comments

6 The curve C has Cartesian equation $x^2+xy+y^2=a$, where a is a positive constant.

(a) Show that the polar equation of C is $r^2 = \frac{2a}{2+\sin 2\theta}$. [3]

$(r\sin\theta)^2 + (r\sin\theta)(r\cos\theta) + (r\cos\theta)^2 = a$

$r^2(\sin^2\theta + \cos^2\theta) + r^2\sin\theta\cos\theta = a$

$r^2 + r^2 \times \frac{1}{2}\sin 2\theta = a$

$r^2(1+\frac{1}{2}\sin 2\theta) = a$ 1

$r^2(2+\sin 2\theta) = 2a$

$r^2 = \frac{2a}{2+\sin 2\theta}$

1 This is a thorough proof.
Mark for (a) = 3 out of 3

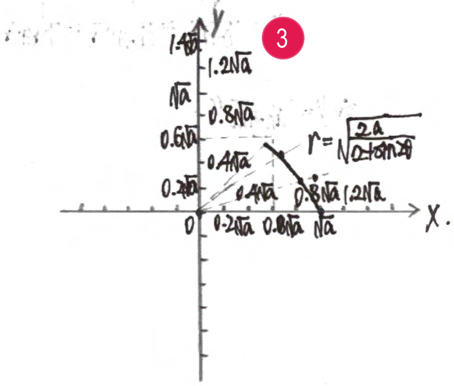
(b) Sketch the part of C for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$. [2]

$\because r^2 = \frac{2a}{2+\sin 2\theta}$

$\therefore r = \sqrt{\frac{2a}{2+\sin 2\theta}}$

x	y
0	$\sqrt{a}$
$\frac{\pi}{12}$	$\sqrt{\frac{4a}{5}} = 0.84\sqrt{a}$
$\frac{\pi}{6}$	$\sqrt{\frac{(6-4\sqrt{3})}{\sqrt{3}}a} = 0.835\sqrt{a}$
$\frac{\pi}{4}$	$\frac{\sqrt{6}}{3}a = 0.816\sqrt{a}$.

2



2 These are correct polar coordinates, but the columns are headed as Cartesians.

3 This needs to be a polar graph, with initial line and pole marked and not x, y axes.
Mark for (b) = 1 out of 2

Example Candidate Response – 2, continued

The region R is enclosed by this part of C, the initial line and the half-line $\theta = \frac{1}{4}\pi$.

(c) It is given that $\sin 2\theta$ may be expressed as $\frac{2\tan\theta}{1+\tan^2\theta}$. Use this result to show that the area of R is

$$\frac{1}{2}a\int_0^{\frac{1}{4}\pi} \frac{1+\tan^2\theta}{1+\tan\theta+\tan^2\theta}\,d\theta$$

and use the substitution $t = \tan\theta$ to find the exact value of this area. [8]

$\frac{1}{2}a\int_0^{\frac{1}{4}\pi} \times$

$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} r^2\,d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{2a}{2+\sin 2\theta}\,d\theta$

$= \frac{1}{2}a\int_0^{\frac{\pi}{4}} \frac{2}{2+\sin 2\theta}\,d\theta$

$= \frac{1}{2}a\int_0^{\frac{\pi}{4}} \frac{2}{2+\frac{2\tan\theta}{1+\tan^2\theta}}\,d\theta$

$= \frac{1}{2}a\int_0^{\frac{\pi}{4}} \frac{2}{\frac{2+2\tan^2\theta+2\tan\theta}{1+\tan^2\theta}}\,d\theta$

4 so Area is $\frac{\ln 3}{6}a$.

$= \frac{1}{2}a\int_0^{\frac{\pi}{4}} \frac{2(1+\tan^2\theta)}{2+2\tan^2\theta+2\tan\theta}\,d\theta$

$= \frac{1}{2}a\int_0^{\frac{\pi}{4}} \frac{1+\tan^2\theta}{1+\tan\theta+\tan^2\theta}\,d\theta = \frac{a}{2}\int_0^{\frac{\pi}{4}} \frac{(1+\tan^2\theta)\cos^2\theta}{1+\tan\theta+\tan^2\theta}\,dt = \frac{a}{2}\int_0^{\frac{\pi}{4}} \frac{1}{1+\tan\theta+\tan^2\theta}\,dt$

As $t=\tan\theta$. $\frac{dt}{d\theta} = \sec^2\theta$ $dt = \sec^2\theta\,d\theta$. $d\theta = \cos^2\theta\,dt$.

$= \frac{1}{\cos^2\theta}$ as $\frac{\pi}{4}=\theta$, $t=1$, $\theta=0$, $t=0$. 5

$= \frac{1}{2}a\int_0^1 \frac{1+t^2}{1+t+t^2} \times \cos^2\theta\,dt. = \frac{1}{2}a\int_0^1 \frac{1}{1+t+t^2}\,dt = \frac{1}{2}a\int_0^1 (1+t+t^2)^{-1}\,dt$

$= \frac{1}{2}a \cancel{[(-1)(1+t+t^2)^{-2} \times (1+2t)]} \left[\frac{\ln(1+t+t^2)}{1+2t}\right]_0^1$

$= \frac{1}{2}a\left[\frac{\ln 3}{3} - \frac{\ln 1}{1}\right]$ 6

$= \frac{1}{2}a \times \frac{\ln 3}{3} = \frac{\ln 3}{6}a.$

Examiner comments

4 The candidates gives a clear demonstration of the answer.

5 This part of the solution is good. The candidate shows clearly how the substitution is being done

6 This is a very common error. The candidate needs to complete the square to make the function fit one of the standard integrals given in the List of formulae (MF19).
Mark for (c) = 4 out of 8

Total mark awarded = 8 out of 13

Question 7

Example Candidate Response – 1

7 The position vectors of the points A, B, C, D are

$$7\mathbf{i}+4\mathbf{j}-\mathbf{k}, \qquad 11\mathbf{i}+3\mathbf{j}, \qquad 2\mathbf{i}+6\mathbf{j}+3\mathbf{k}, \qquad 2\mathbf{i}+7\mathbf{j}+\lambda\mathbf{k}$$

respectively.

(a) Given that the shortest distance between the line AB and the line CD is 3, show that $\lambda^2-5\lambda+4=0$. [7]

1

$\vec{AB}=\vec{OB}-\vec{OA}=\begin{pmatrix}11\\3\\0\end{pmatrix}-\begin{pmatrix}7\\4\\-1\end{pmatrix}=\begin{pmatrix}4\\-1\\1\end{pmatrix}$

$\vec{CD}=\vec{OD}-\vec{OC}=\begin{pmatrix}2\\7\\\lambda\end{pmatrix}-\begin{pmatrix}2\\6\\3\end{pmatrix}=\begin{pmatrix}\cancel{6}0\\1\\\lambda-3\end{pmatrix}$

$\vec{n}=\begin{pmatrix}4\\-1\\1\end{pmatrix}\times\begin{pmatrix}\cancel{6}0\\1\\\lambda-3\end{pmatrix}$

$=\begin{pmatrix}-\lambda+2\\-4\lambda+12\\4\end{pmatrix}$

2

$\begin{pmatrix}-\lambda+2\\-4\lambda+12\\4\end{pmatrix}\cdot\begin{pmatrix}7\\4\\-1\end{pmatrix}=-7\lambda+14-16\lambda+48-4$

$=-23\lambda+\cancel{30}\,58$

$\dfrac{-23\lambda+\cancel{30}\,58}{\sqrt{(\lambda+2)^2+(4\lambda+12)^2+\cancel{16}(4)^2}}=3$

$\cancel{(-23\lambda+58)^2=9(\lambda^2+4\lambda+4+\lambda^2-24\lambda+144+16)}$

$529\lambda^2\cancel{-2668\lambda+3364=9\lambda^2+18\lambda^2-\overset{252}{180}\lambda+1476}$

$\cancel{511\lambda^2-2488\lambda+1888=0}$

2920

3

$\cancel{(-23\lambda+58)^2=9(\lambda^2-4\lambda+4+16\lambda^2-96\lambda+160)}$

$\cancel{529\lambda^2-2668\lambda+3364=153\lambda^2-900\lambda+1476}$

$\cancel{376\lambda^2-1768\lambda+1888=0}$

$(-23\lambda+58)^2=9[(-\lambda+2)^2+(-4\lambda+12)^2+(4^2)]$

$376\lambda^2-1880\lambda+1504=0$

4

$\lambda^2-5\lambda+4=0$

Examiner comments

1 Vectors are correctly named which helps when reading the work.

2 Correct direction vector for common perpendicular.

3 This is the position vector of A, so they are using a wrong formula.

4 Although the candidate is using a wrong formula, their attempt to reach the given answer by squaring and simplifying to a three-term quadratic scores the last method mark.
Mark for (a) = 4 out of 7

Example Candidate Response – 1, continued	Examiner comments
Let Π_1 be the plane ABD when $\lambda = 1$. Let Π_2 be the plane ABD when $\lambda = 4$. **(b)** **(i)** Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$. $r = \begin{pmatrix}7\\4\\-1\end{pmatrix} + s\begin{pmatrix}4\\-1\\1\end{pmatrix} + t\begin{pmatrix}0\\1\\-2\end{pmatrix}$ 5	5 The candidate needs to show this vector is an attempt at a vector lying in the plane. It could be named, or the subtraction could be shown. Mark for (b)(i) = 0 out of 2
(ii) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. $r = \begin{pmatrix}7\\4\\-1\end{pmatrix} + s\begin{pmatrix}4\\-1\\1\end{pmatrix} + t\begin{pmatrix}0\\1\\1\end{pmatrix}$ 6 $\begin{vmatrix} i & j & k \\ 4 & -1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2x - 4y + 4z = \begin{pmatrix}7\\4\\-1\end{pmatrix} \cdot \begin{pmatrix}-2\\-4\\4\end{pmatrix}$ 7 $-2x - 4y + 4z = -14 - 16 - 4$ 8 $x + 2y - 2z = 17$	6 This unnamed vector does not lie in the plane. To use the equation in this form both vectors must lie in the plane. 7 The candidate is taking the cross-product correctly for *their* direction vectors. 8 The candidate uses a correct point to complete the equation of the plane. Mark for (b)(ii) = 2 out of 4
(c) Find the acute angle between Π_1 and Π_2. $\vec{n}_1 = \begin{vmatrix} i & j & k \\ 4 & -1 & 1 \\ 0 & 1 & -2 \end{vmatrix} = x + 8y + 4z$ 9 $\begin{pmatrix}1\\8\\4\end{pmatrix} \cdot \begin{pmatrix}1\\2\\-2\end{pmatrix} = \sqrt{1+8^2+4^2} \cdot \sqrt{1+4+4}\, \cos\beta$ $\cos\beta = \frac{9}{27}$ 10 $\cos\beta = \frac{1}{3}$ $\beta = 70.5°$	9 Correct cross-product for normal to *their* plane 1. 10 The candidate uses the Scalar product correctly to find acute angle. Mark for (c) = 2 out of 5 **Total mark awarded = 8 out of 18**

Example Candidate Response – 2	Examiner comments

7 The position vectors of the points A, B, C, D are

$$7\mathbf{i}+4\mathbf{j}-\mathbf{k}, \qquad 11\mathbf{i}+3\mathbf{j}, \qquad 2\mathbf{i}+6\mathbf{j}+3\mathbf{k}, \qquad 2\mathbf{i}+7\mathbf{j}+\lambda\mathbf{k}$$

respectively.

(a) Given that the shortest distance between the line AB and the line CD is 3, show that $\lambda^2-5\lambda+4=0$. [7]

$\overrightarrow{AB} = 11i+3j-7i-4j+k = 4i-j+k$

line AB = $7i+4j-k+t(4i-j+k)$ ①

$\overrightarrow{CD} = 2i+7j+\lambda k-2i-6j-3k = j+(\lambda-3)k$ ②

line CD = $2i+6j+3k+t(j+(\lambda-3)k)$

~~$(2i+6j+3k-7i-4j+k)\cdot($~~

$$\begin{vmatrix} i & j & k \\ 4 & -1 & 1 \\ 0 & 1 & \lambda-3 \end{vmatrix}$$

$(3-\lambda-1)i-(4\lambda-12)j+4k$

$(2-\lambda)i-(4\lambda-12)j+4k$

$$\frac{(2i+6j+3k-7i-4j+k)\cdot((2-\lambda)i-(4\lambda-12)j+4k)}{(2-\lambda)i-(4\lambda-12)j+4k} = 3$$

$$\frac{(-5i+2j+4k)((2-\lambda)i-(4\lambda-12)j+4k)}{|(2-\lambda)i-(4\lambda-12)j+4k|} = 3.$$

$$\frac{-10+5\lambda+24-8\lambda+16}{\sqrt{(2-\lambda)^2+(12-4\lambda)^2+16}} = 3$$

$$\frac{(-3\lambda+30)^2}{(2-\lambda)^2+(12-4\lambda)^2+16} = 9$$ ③

$9\lambda^2-180\lambda+900 = 9(4-4\lambda+\lambda^2+144-96\lambda+\lambda^2+16)$

$9\lambda^2-180\lambda+900 = 18\lambda^2-900\lambda+1476$

$9\lambda^2-720\lambda+576=0$

$\lambda^2-80\lambda+64=0$

Examiner comments

① The equation is not needed but it should really begin '$r =$'.

② The vectors are correctly named. Candidates who name their vectors find it easier to check their arithmetic and their method.

③ The candidate's method is correct, but they have made one error in expanding brackets.
Mark for (a) = 6 out of 7

Example Candidate Response – 2, continued	Examiner comments
Let Π_1 be the plane ABD when $\lambda = 1$. Let Π_2 be the plane ABD when $\lambda = 4$. **(b) (i)** Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$. $\vec{AB}: 4i - j + k$ $\vec{AD}: -5i + 3j + 2k$ (4) $\Pi_1 = 7i + 4j - k + s(4i - j + k) + t(-5i + 3j + 2k)$ (5)	(4) It is good to see the vectors named. (5) This needs $r =$ to be the equation of a plane. Mark for (b)(i) = 1 out of 2
(ii) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. $\vec{AB}: 4i - j + k$ $\vec{AD}: -5i + 3j + 5k$ $\begin{vmatrix} i & j & k \\ 4 & -1 & 1 \\ -5 & 3 & 5 \end{vmatrix}$ (6) $(-5-3)i - (20+5)j + (12+5)k$ $= -8i - 25j + 17k$ (7) $(7i + 4j - k)\cdot(-8i - 25j + 17k) = -173$ ~~$-8x - 25y + 17z = -173$~~ $-8x - 25y + 17z = -173$	(6) It is very common to make a sign error in this cross-product. (7) A suitable point is used correctly to find the equation of the plane. Mark for (b)(ii) = 2 out of 4
(c) Find the acute angle between Π_1 and Π_2. $\begin{vmatrix} i & j & k \\ 4 & -1 & 1 \\ -5 & 3 & 2 \end{vmatrix}$ (8) $(-2-3)i - (8+5)j + (8+5)k = -5i - 13j + 13k$ $\dfrac{-5\times(-8) + (-13)\times(-25) + (13\times17)}{\sqrt{5^2+13^2+13^2} \times \sqrt{8^2+25^2+17^2}} = \cos\theta$ (9) $\cos\theta = 0.98$ $\theta = 10.42°$ $180 - 10.42 = 169.6°$ (10)	(8) Again, the candidate has a sign error in the cross-product so accuracy marks cannot be awarded. (9) Dot product of *their* normal vectors are being used to find the angle. (10) It is a common error for candidates to subtract from 180 or 90 degrees. The question asks for the acute angle as the final answer. Mark for (c) = 2 out of 5 **Total mark awarded = 11 out of 18**

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