Cambridge Assessment International Education

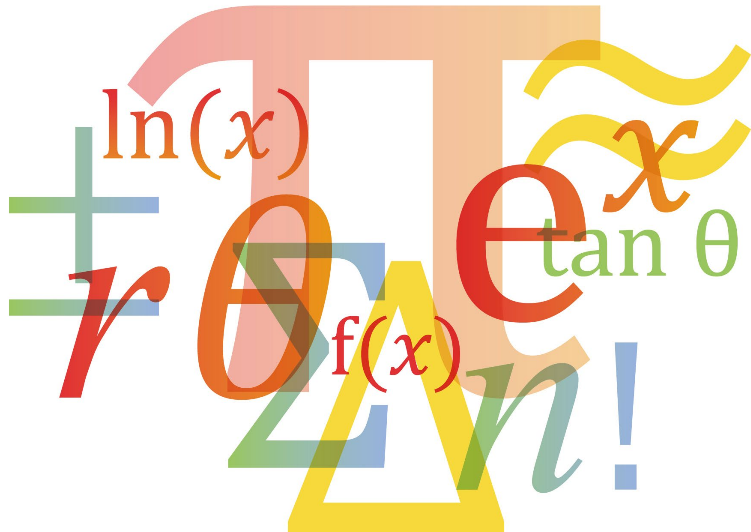
Scheme of Work – Paper 3

Cambridge International AS & A Level

Mathematics 9709

Pure Mathematics 3

For examination from 2020



Version 2

Cambridge Pathway

Contents

Introduction

The Cambridge International AS & A Level Mathematics 9709 scheme of work has been designed to support you in your teaching and lesson planning. The Scheme of Work has been separated into six documents, one for each content section: Pure Mathematics 1, Pure Mathematics 2, Pure Mathematics 3, Mechanics, Probability & Statistics 1 and Probability & Statistics 2. This document relates only to **Pure Mathematics 3**.

Making full use of this scheme of work will help you to improve both your teaching and your learners' potential. It is important to have a scheme of work in place in order for you to guarantee that the syllabus is covered fully. You can choose what approach to take and you know the nature of your institution and the levels of ability of your learners. What follows is just one possible approach you could take and you should always check the syllabus for the content of your course.

Suggestions for independent study **(I)** and formative assessment **(F)** are also included. Opportunities for differentiation are indicated as **Extension activities**; there is the potential for differentiation by resource, grouping, expected level of outcome, and degree of support by teacher, throughout the scheme of work. Timings for activities and feedback are left to the judgement of the teacher, according to the level of the learners and size of the class. Length of time allocated to a task is another possible area for differentiation.

Key concepts

This scheme of work is underpinned by the assumption that mathematics involves the application of logical methodologies, problem solving and the recognition of patterns as well as the application of these approaches to mathematical modelling. The key concepts are highlighted as a separate item in the new syllabus and you should be aware that learners will be assessed on their direct knowledge and understanding of the same. Learners should be able to describe and explain the key concepts as well as demonstrate their ability to apply them to novel situations and evaluate them. The key concepts for Cambridge International AS & A Level Mathematics are:

Key Concept – Problem solving

Key Concept – Communication

Key Concept – Mathematical modelling

See the syllabus for detailed descriptions of each Key Concept.

Guided learning hours

Guided learning hours give an indication of the amount of contact time teachers need to have with learners to deliver a particular course. Our syllabuses are designed around 180 hours for Cambridge International AS Level, and 360 hours for Cambridge International A Level. The number of hours may vary depending on local practice and your learners' previous experience of the subject. The table below gives some guidance about how many hours are recommended for each topic.

It is recommended that you spend about 115 hours altogether teaching the content of Pure Mathematics 3, covering both the AS and A Level course.

Topic	Suggested teaching time (hours)	Suggested teaching order
3.1 Algebra	It is recommended that this should take about 10 hours.	1
3.2 Logarithmic and exponential functions	It is recommended that this should take about 14 hours.	2
3.3 Trigonometry	It is recommended that this should take about 12 hours.	3
3.4 Differentiation	It is recommended that this should take about 18 hours.	4
3.5 Integration	It is recommended that this should take about 16 hours.	5
3.6 Numerical solution of equations	It is recommended that this should take about 10 hours.	6
3.7 Vectors	It is recommended that this should take about 12 hours.	7
3.8 Differential equations	It is recommended that this should take about 8 hours.	8
3.9 Complex numbers	It is recommended that this should take about 15 hours.	9

Prior knowledge

Knowledge of the content of Paper 1: Pure Mathematics 1 is assumed, and candidates may be required to demonstrate such knowledge in answering questions.

Resources

You can find the endorsed resources to support Cambridge International AS & A Level Mathematics on the Published resources tab of the syllabus page on our public website here.

Endorsed textbooks have been written to be closely aligned to the syllabus they support, and have been through a detailed quality assurance process. All textbooks endorsed by Cambridge International for this syllabus are the ideal resource to be used alongside this scheme of work as they cover each learning objective. In addition to reading the syllabus, teachers should refer to the specimen assessment materials.

School Support Hub

The School Support Hub www.cambridgeinternational.org/support is a secure online resource bank and community forum for Cambridge teachers, where you can download specimen and past question papers, mark schemes and other resources. We also offer online and face-to-face training; details of forthcoming training opportunities are posted online. This scheme of work is available as PDF and an editable version in Microsoft Word format; both are available on the School Support Hub at www.cambridgeinternational.org/support. If you are unable to use Microsoft Word you can download Open Office free of charge from www.openoffice.org

Websites

This scheme of work includes website links providing direct access to internet resources. Cambridge Assessment International Education is not responsible for the accuracy or content of information contained in these sites. The inclusion of a link to an external website should not be understood to be an endorsement of that website or the site's owners (or their products/services).

The website pages referenced in this scheme of work were selected when the scheme of work was produced. Other aspects of the sites were not checked and only the particular resources are recommended.

How to get the most out of this scheme of work – integrating syllabus content, skills and teaching strategies

We have written this scheme of work for the Cambridge International AS & A Level Mathematics 9709 syllabus and it provides some ideas and suggestions of how to cover the content of the syllabus. We have designed the following features to help guide you through your course.

Learning objectives help your learners by making it clear the knowledge they are trying to build. Pass these on to your learners by expressing them as 'We are learning to / about…'.

Suggested teaching activities give you lots of ideas about how you can present learners with new information without teacher talk or videos. Try more active methods which get your learners motivated and practising new skills.

Learning objectives	Suggested teaching activities
$y = e^{kx}$ for both positive and negative values of k	is the same as itself. With a suitable graph plotter you can demonstrate that the gradient function of $y = e^x$ i **Extension activity:** There are other, formal, approaches that you could use with more capable learners. F you could consider compound interest and the limit of the series $\left(1+\frac{1}{n}\right)^n$ as shown at: www.mathsisfun.com/numbers/e-eulers-number.html Encourage learners to obtain the logarithmic form of the statement $e^x = a$ and so introduce them to natural logarithms. Building on the work done in Pure Mathematics 1.2 'Functions', develop this into the inverse re between e^x and $\ln x$ and demonstrate the inverses on a graph plotter. An interactive exercise covering thi relationship is at: http://hotmath.com/help/gt/genericalg2/section_8_5.html **(I)**
• use logarithms to solve equations and inequalities in which the unknown appears in indices, e.g. $2^x < 5$, $3 \times 2^{3x-1} < 5$, $3^{x+1} = 4^{2x-1}$	As a whole class exercise, work through some examples of increasing difficulty, using carefully directed questioning to work through the solutions. Textbooks will include many examples of this type of question and the interactive exercise at the link above includes some too. Demonstrate examples using inequalities, with learners finding critical values first and then deducing the set of solutions. It is helpful to highlight to learners the sign of $\ln x$ for $0 < x \le 1$, perhaps through an example where the inequality reverses

Independent study (I) gives your learners the opportunity to develop their own ideas and understanding with direct input from you.

Extension activities provide your more able learners with further challenge beyond the basic content of the course. Innovation and independent learning are the basis of these activities.

Past papers, specimen papers and **mark schemes** are available for you to download at: www.cambridgeinternational.org/support

Using these resources with your learners allows you to check their progress and give them confidence and understanding.

Formative assessment (F) is ongoing assessment which informs you about the progress of your learners. Don't forget to leave time to review what your learners have learnt, you could try question and answer, tests, quizzes, 'mind maps', or 'concept maps'. These kinds of activities can be found in the scheme of work.

3.1 Algebra

Learning objectives	Suggested teaching activities
• understand the meaning of $\|x\|$, sketch the graph of $y = \|ax + b\|$ and use relations such as $\|a\| = \|b\| \Leftrightarrow a^2 = b^2$ and $\|x - a\| < b \Leftrightarrow a - b < x < a + b$ when solving equations and inequalities, e.g. $\|3x - 2\| = \|2x + 7\|$, $2x + 5 < \|x + 1\|$; graphs of $y = \|f(x)\|$ and $y = f(\|x\|)$ for non-linear functions f are not included;	To introduce the notation, start with a numerical value, e.g. –5, and discuss the meaning of $\|-5\|$. Help learners to deduce the results $\|a\| = \|b\| \Leftrightarrow a^2 = b^2$ and $\|x - a\| < b \Leftrightarrow a - b < x < a + b$ as part of a class discussion. Some really useful resources are at www.tes.co.uk/teaching-resource/a-level-maths-c2-modulus-function-worksheets-6146818: • 'Modulus Function Introduction' provides a worksheet for learners to complete. **(I)** • 'Solving Modulus Equations and Inequalities' could be used for consolidation/practice. **(I)** • 'Modulus Transformations' provides practice at sketching graphs involving a modulus. Demonstrate some initially to learners using a graph plotter. **(I)** • 'Alternative Methods for Solving Modulus Equations' is a worksheet which helps learners to explore the different ways of solving this type of equation. **(I)** Learners investigate the connection between the shape of the graph of $y = ax + b$ and the shape of the graph of $y = \|ax + b\|$ by plotting a range of these using graphing software. **(I)** The graphs of various modulus functions are at: www.mathsmutt.co.uk/files/mod.htm Suitable past/specimen papers for practice and/or formative assessment include **(I)(F)**:
• divide a polynomial, of degree not exceeding 4, by a linear or quadratic polynomial, and identify the quotient and remainder (which may be zero)	There are several different methods of polynomial division including inspection, the table method, and long division. This PowerPoint presentation introduces all three methods for factorising cubics. You can use the methods for any polynomial and also for division that results in a remainder: www.furthermaths.org.uk/files/sample/files/edx/Factorising_cubics.ppt When teaching any of the methods, start with a numerical example to remind learners of the thought process they need, and use this to introduce the terms 'quotient' and 'remainder'. For example, $54763 \div 8$ leads to a quotient of 6845 and a remainder of 3. Continue with a simple algebraic example $(x^2 + 4x + 1) \div (x + 2)$ which leads to a quotient of $x + 2$ and a remainder of –3. You will probably need to show learners further examples involving more complex polynomials before they practise on their own.

Learning objectives	Suggested teaching activities
	Ideas on possible approaches you can take for long division are at: www.khanacademy.org/math/algebra2/polynomial_and_rational/dividing_polynomials/v/dividing-polynomials-with-remainders and www.mathsisfun.com/algebra/polynomials-division-long.html A worksheet of examples for practising any of the methods for division is at: www.mathworksheetsgo.com/sheets/algebra-2/polynomials/dividing-polynomials-worksheet.php **(I)** There is another approach known as synthetic division but learners have to be careful when using it, especially when factorising. Textbooks will have many useful questions for learners to practise.
• use the factor theorem and the remainder theorem, e.g. to find factors and remainders, solve polynomial equations or evaluate unknown coefficients, including factors of the form $(ax + b)$ in which the coefficient of x is not unity, and including calculation of remainders	Summarise the work already done on polynomial division to show that $p(x)$ = (divisor $\times$ quotient) + remainder. Show that algebraic division can often be avoided in questions by substituting into $p(x)$ the value of x that makes the divisor zero (e.g. substituting 3 if the divisor is $x - 3$ and calculating p(3) to find the remainder). Show that the factor theorem is a special case of the remainder theorem when the remainder is zero. A good approach of this type which you could use with a whole class is at: www.mathsisfun.com/algebra/polynomials-remainder-factor.html Show examples involving finding factors, solving polynomial equations and evaluating unknown coefficients to the whole class, questioning learners individually throughout. Remind learners that they should show all their working as the use of a calculator for finding solutions to polynomial equations will not be accepted in an exam. A useful worksheet which covers basic use of the remainder theorem and evaluating unknown coefficients (log in for free download) is at: www.tes.co.uk/teaching-resource/worksheet-on-the-remainder-theorem-6140286 **(I)** More examples on the remainder theorem and on solving polynomial equations are at: www.mash.dept.shef.ac.uk/Resources/A26remainder.pdf
• recall an appropriate form for expressing rational functions in partial fractions, and carry out the decomposition, in cases where the	Examples of the three main types of partial fraction are here (log in for free download): www.tes.com/teaching-resource/partial-fractions-examples-6140352 Some worked examples and 10 practice questions for learners to try (at the end of the document) are at:

Learning objectives	Suggested teaching activities
denominator is no more complicated than: - $(ax + b)(cx + d)(ex + f)$ - $(ax + b)(cx + d)^2$ - $(ax + b)(cx^2 + d)$ excluding cases where the degree of the numerator exceeds that of the denominator	www.mathsisfun.com/algebra/partial-fractions.html Textbooks will also contain many examples for learners to practise. In many questions, the first part will involve breaking down rational functions into partial fractions and later parts will use partial fractions with another mathematical technique such as binomial expansion, integration or solving differential equations. Set learners questions involving these topics when they have covered them.
• use the expansion of $(1 + x)^n$, where n is a rational number and $\|x\|<1$; finding a general term in an expansion is not included; adapting the standard series to expand e.g. $\left(2-\frac{1}{2}x\right)^{-1}$ is included, and determining the set of values of x for which the expansion is valid in such cases is also included	Learners have already met the binomial expansion in Pure Mathematics 1.6 'Series' so, to check their understanding, set them some preparatory questions on basic binomial expansions using the formula $(a+b)^n$, where n is a positive integer. **(I)** Ask learners to work out the first few terms of the expansion of $(1+x)^n$ from the formula for expanding $(a+b)^n$, to obtain $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3...$ This is now in a useful form for introducing negative and fractional powers. This tutorial shows that you need the condition $\|x\|<1$ for negative powers because they generate an infinite series. The first few terms are only a good approximation if the values of x meet this condition and the series converges: www.examsolutions.net/tutorials/binomial-expansion-validity/?level=International&board=CIE&module=P3&topic=1308 This link uses an example with $n = \frac{1}{2}$ and has an interesting graphical display of the approximation. www.intmath.com/series-binomial-theorem/4-binomial-theorem.php Textbooks will include many examples for learners to practise expanding and finding the range of values for which each expansion is valid.**(I)** Demonstrate to learners how to rewrite examples of the type $\left(2-\frac{1}{2}x\right)^{-1}$ as $\frac{1}{2}\left(1-\frac{x}{4}\right)^{-1}$ so that they can go on to expand them.

Past and specimen papers
Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support **(F)** 9709 Mathematics 2020 Specimen Paper 2

3.2 Logarithmic and exponential functions

Learning objectives	Suggested teaching activities
• understand the relationship between logarithms and indices, and use the laws of logarithms (excluding change of base)	Start by defining the terms 'logarithm' and 'exponential', linking to the concept of indices. To help learners understand a statement such as $\log_a x = b$, describe it to them in words such as "What power of *a* is *x*? Answer: *b*" An introduction with animation showing the relationship between logarithms and exponentials is at: www.purplemath.com/modules/logs.htm . Learners should practise converting expressions from logarithmic to exponential form and from exponential form to logarithmic. Most textbooks will have plenty of examples of this type. A useful worksheet (includes the laws of logarithms) is at: maths.mq.edu.au/numeracy/web_mums/module2/Worksheet27/module2.pdf **(I)** To introduce the laws of logarithms, start with statements $\log_a x = b$ and $\log_a y = c$. Use targeted questioning to encourage learners to write the exponential forms of these statements and reach the conclusion that $a^{b+c} = xy$, rewriting this in logarithmic form to obtain $\log_a xy = \log_a x + \log_a y$. Ask learners to obtain the other two laws in a similar way. Learners will then need to practise applying these laws. Eight files of notes, worksheets and revision (log in for free download) are at: www.tes.co.uk/teaching-resource/a-level-maths-logarithms-worksheets-and-revision-6146791 **(I)** An additional resource which demonstrates the above approach is at: www.mathsisfun.com/algebra/exponents-logarithms.html
• understand the definition and properties of e^x and ln*x*, including their relationship as inverse functions and their graphs; including knowledge of the graph of $y = e^{kx}$ for both positive and negative values of *k*	You can introduce the exponential function e^x in various ways. One approach is to use a graph plotter to show learners the graphs of various exponential functions, e.g. $y = 2^x$, $y = 3^x$, $y = 5^x$. Develop the idea of a particular exponential function that lies between $y = 2^x$ and $y = 3^x$, such that its gradient function is the same as itself. With a suitable graph plotter you can demonstrate that the gradient function of $y = e^x$ is e^x.

Learning objectives	Suggested teaching activities
	Extension activity: There are other, formal, approaches that you could use with more capable learners. For example you could consider compound interest and the limit of the series $\left(1+\frac{1}{n}\right)^n$ as shown at: www.mathsisfun.com/numbers/e-eulers-number.html Encourage learners to obtain the logarithmic form of the statement $e^x = a$ and so introduce them to natural logarithms. Building on the work done in Pure Mathematics 1.2 'Functions', develop this into the inverse relationship between e^x and $\ln x$ and demonstrate the inverses on a graph plotter. An interactive exercise covering this relationship is at: http://hotmath.com/help/gt/genericalg2/section_8_5.html **(I)**
• use logarithms to solve equations and inequalities in which the unknown appears in indices, e.g. $2^x < 5$, $3 \times 2^{3x-1} < 5$, $3^{x+1} = 4^{2x-1}$	As a whole class exercise, work through some examples of increasing difficulty, using carefully directed questioning to work through the solutions. Textbooks will include many examples of this type of question and the interactive exercise at the link above includes some too. Demonstrate examples using inequalities, with learners finding critical values first and then deducing the set of solutions. It is helpful to highlight to learners the sign of $\ln x$ for $0 < x \leq 1$, perhaps through an example where the inequality reverses.
• use logarithms to transform a given relationship to linear form, and hence determine unknown constants by considering the gradient and/or intercept, e.g. $y = kx^n$ gives $\ln y = \ln k + n \ln x$ which is linear in $\ln x$ and $\ln y$; $y = k(a^x)$ gives $\ln y = \ln k + x \ln a$ which is linear in x and $\ln y$	If you relate this technique to practical situations, this will help learners when they need to use it in their scientific subjects. Common forms of equation are $y = Ab^x$ and $y = Ax^b$. Learners will need to be able to write these equations in logarithmic form and hence relate them to the equation of a straight line. Sometimes the variables will be letters other than *x* and *y* so learners need to spot the form of the equation in order to distinguish the variables from the constants. A useful summary for dealing with situations involving $y = Ax^b$ is at: http://mathbench.umd.edu/modules/misc_scaling/page11.htm Either work through this with learners in class or they could study it independently. **(I)** Use a similar approach for equations of the type $y = Ab^x$. Work through such an example in class, making use of a graph plotter to demonstrate the straight line obtained. Textbooks will provide learners with many useful practice questions. For variety, try to choose examples which involve variables other than *x* and *y*. Often, learners are asked to work from a given graph in straight line form. Common errors involve learners considering *y* values rather than $\ln y$ values, so practising questions will help to

Learning objectives	Suggested teaching activities
	avoid such errors. The Paper 2 past exam papers have examples of this type. To help reinforce this point, split learners into groups or pairs and ask each of them to prepare a question. A simple way to do this is for learners to 'work backwards' from a logarithmic relationship e.g. $P = At^b$. Each group chooses values for A and b, works out the coordinates of two pairs of coordinates and draws an appropriate straight line graph. Learners circulate their graphs around the other groups who then identify the logarithmic equations used to draw the graphs.

3.3 Trigonometry

Subject content	Suggested teaching activities
• understand the relationship of the secant, cosecant and cotangent functions to cosine, sine and tangent, and use properties and graphs of all six trigonometric functions for angles of any magnitude	Start by defining the secant, cosecant and cotangent functions. Learners should know the graphs of the sine, cosine and tangent functions so, as a group or individual task, ask them to think what the graphs of the secant, cosecant and cotangent functions look like. For example, give them the graph of y = sin x (from –360° to 720°) and ask them to sketch y = cosec x on the same axes. They then check using a graph plotter. Use a similar graphical approach for y = sec x and y = cot x.
• use trigonometrical identities for the simplification and exact evaluation of expressions, e.g. simplifying cos(x – 30°) – 3sin(x – 60°), and in the course of solving equations, e.g. solving tan θ + cot θ = 4, 2 sec$^2\theta$ – tan θ = 5, 3 cosθ + 2sin θ = 1 and select an identity or identities appropriate to the context, showing familiarity in particular with the use of: - sec$^2\theta \equiv 1 + \tan^2\theta$ and cosec$^2\theta \equiv 1 + \cot^2\theta$ - the expansions of sin(A ± B), cos(A ± B) and tan(A ± B) - the formulae for sin 2A, cos 2A and tan 2A - the expression of *a sin θ + b cos θ* in the forms R sin(θ ± α) and R cos(θ ± α),	Start with the identity $\sin^2\theta + \cos^2\theta \equiv 1$ (which learners know already) and ask what they find when: (a) they divide each term in this identity by $\cos^2\theta$ and (b) they divide each term in the original identity by $\sin^2\theta$. A matching exercise and a worksheet for learners to complete as consolidation and practice are at: www.tes.com/teaching-resource/a-level-maths-reciprocal-trig-functions-worksheet-6146865 **(I)** Learners will need plenty of practice at simplifying trigonometric expressions and using the identities, particularly questions of the 'Show that' or 'Prove that' type. The best strategy is to start with one side of the expression (usually the left hand side) and manipulate it using the identities covered so far. Textbooks will include some practice questions. An exercise on simplification is at: http://worksheets.tutorvista.com/proving-trigonometric-identities-worksheet.html **(I)** An exercise on proof is at: https://people.math.osu.edu/maharry.1/150Au2011/TrigIdentities.pdf **(I)** Learners will need to be able to use the identities to solve equations in degrees or radians, and textbooks will contain useful exercises on this. Learners will also need to practise manipulating expressions to obtain an equation (usually quadratic) in terms of one trigonometric ratio e.g. $2\sec^2\theta - 3 + \tan\theta = 0$ will simplify to $2\tan^2\theta + \tan\theta - 1 = 0$ which factorises. For the compound angle (addition) formulae, work through an example of how one formula is derived, perhaps as a

Subject content	Suggested teaching activities
	whole class exercise. A video proof is at: www.youtube.com/watch?v=a0LvqflQMx4 The proof of one formula is covered in a similar way at: www.trans4mind.com/personal_development/mathematics/trigonometry/compoundAngleProofs.htm#mozTocId169602 **Extension activity:** Ask learners to work out the proofs of some of the other formulae. Alternatively, start by giving learners the challenge of deriving the compound angle formulae graphically using this interesting investigation: www.tes.co.uk/teaching-resource/the-compound-angle-formulae-lesson-worksheet-6056103 Proving the formulae may come more easily to learners once they are more familiar with them. When learners are competent with the compound angle formulae, ask them to derive the double-angle formulae. They will need to find all possible variants of the formula for $\cos 2\theta$ as well as rearranging them to $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ and $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ for use in other applications such as integration. Textbooks include many useful practice exercises on solving equations using the compound and double-angle formulae. Make sure that learners are proficient at using radians as well as degrees. **(I)** A clear summary of how to deal with expressions of the type $a\sin\theta + b\cos\theta$ is at: www.intmath.com/analytic-trigonometry/6-express-sin-sum-angles.php Start with an example e.g. $3\sin\theta + 4\cos\theta$ and show that it may be written in the form $5\sin(\theta + 53.13^\circ)$. This can also be verified using a graph plotter: show learners the graph of $y = 3\sin\theta + 4\cos\theta$ and, with a discussion on transformations, encourage learners to write this expression in a different way. They can check the result by plotting the equivalent expression and seeing that it gives the same graph. Next ask learners to find the maximum and minimum values of the expression and the values of θ at which they occur. (Discourage the use of calculus for questions of this type.) Textbooks include many examples of writing equivalent expressions, solving equations and finding maximum and minimum values. Learners need to be proficient at using radians as well as degrees. **(I)**

4 Differentiation

Subject content	Suggested teaching activities
• use the derivatives of e^x, ln x, sin x, cos x, tan x, $\tan^{-1}x$, together with constant multiples, sums, differences and composites; derivatives of $\sin^{-1}x$ and $\cos^{-1}x$ not required	A good approach to teaching this section is to use a whole class approach and targeted questioning of learners. For the function $y = e^x$, learners already know that the gradient function is e^x. Build on this by differentiating other functions such as $y = e^{mx}$, $y = e^{f(x)}$, making use of the chain rule where appropriate. To differentiate $y = \ln x$, write $x = e^y$, so $\frac{dx}{dy} = e^y$ and you can obtain the result $\frac{dy}{dx} = \frac{1}{x}$. Using the chain rule, you can generalise to expressions of the form $\frac{d}{dx}\left(\ln f(x)\right) = \frac{f'(x)}{f(x)}$. Textbooks will have exercises for learners to practise. **(I)** To obtain the derivatives of $\sin x$ and $\cos x$, consider the gradient of a chord from the origin to a point (h, sin h) on the curve y = sin x. Ask learners to calculate the gradient sin h / h (where h is 0.1 then 0.01 then 0.001) and use this to deduce the gradient at x = 0. They then deduce the gradient at other key points on the graph, for instance $x = 0, \pi/2, \pi, 3\pi/2, 2\pi$, use their values to plot the gradient function on a graph of y = sin x and name the graph obtained. Show them that a similar approach will give them the gradient function for y = cos x. You can find this method in many textbooks. It is also covered at: www.mathcentre.ac.uk/resources/uploaded/mc-ty-sincos-2009-1.pdf **Extension activity:** The resource above also covers differentiation from first principles which is suitable as an extension for the more capable learner. Encourage learners to obtain results for the derivatives of sin mx, cos mx, sin f(x) and cos f(x) during a class discussion, making use of the chain rule. Leave the differentiation of y = tan x until the quotient rule has been covered. The differentiation of $y = \tan^{-1}x$ is covered in a video at: www.khanacademy.org/math. Search for 'derivative of inverse tangent'. Learners will need to have covered implicit differentiation and be confident working with the chain rule before covering this. This is an opportunity for a flipped learning task. **(I)**

Subject content	Suggested teaching activities
	Many textbooks will have exercises for learners to practice. **(I)**
• differentiate products and quotients, e.g. $\frac{2x-4}{3x+2}$, $x^2\ln x$, xe^{1-x^2}	Derive the product and quotient rules as a whole class exercise so that learners (especially the more able) can understand the formulae more thoroughly. There is a proof using function notation at: http://nrich.maths.org/10086. Alternatively, write the product as uv (where u and v are functions of x) then consider increasing the area of a rectangle uv to $(u + \delta u)(v + \delta v)$. Expanding the brackets, writing every term over δx and considering the limit as δx -> 0 leads to the product rule. Three files of examples and worksheets on differentiation of products are at: www.tes.co.uk/teaching-resource/product-and-quotient-rules-6146838 **(I)** Appropriate textbooks will have further examples. Try to introduce a variety of different types of functions (such as those in the previous section) and encourage learners to simplify their answers. Set learners the task of deriving the quotient rule by differentiating $y=\frac{u}{v}$, where u and v are functions of x, as a product $y = uv^{-1}$, using the product rule. Ask learners to differentiate $y = \tan x$ using the quotient rule. Three files which include examples/worksheets on differentiation of quotients (log in for free download) are at: www.tes.co.uk/teaching-resource/product-and-quotient-rules-6146838 **(I)** Appropriate textbooks will have further examples. Try to introduce a variety of different types of functions (such as those in the previous section) and encourage learners to simplify their answers.
• find and use the first derivative of a function which is defined parametrically or implicitly, e.g. $x = t - e^{2t}$, $y = t + e^{2t}$, e.g. $x^2 + y^2 = xy + 7$, including use in problems involving tangents and normals	Introduce the idea of parametric equations to learners by asking them to imagine two cars moving towards each other along different straight lines on the x-y plane. You know their lines will intersect but how do you know if the cars will collide or miss each other? You need to consider a third parameter (e.g. time), and express both x and y in terms of this parameter, in order to say whether or not there will be a collision. Then show learners some simple examples, e.g. $x = 2t$, $y = 3t^2 + 5$ and eliminate t to obtain the Cartesian form of the curve. A graph plotter may be useful.

Subject content	Suggested teaching activities
	Show that the gradient function may be obtained using the derivatives $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ together with the chain rule. Extend the work to include parametric equations involving trigonometric functions e.g. $y = 3\cos 2\theta,\ x = 4\sin\theta$ to help learners to consolidate their knowledge of trigonometric identities and differentiation of trigonometric functions. A clear and thorough treatment of the topic with worked examples (see 17.1 Cartesian and parametric equations of a curve and 17.4 Parametric differentiation) is at: www.cimt.org.uk/projects/mepres/alevel/pure_ch17.pdf A good overview of the topic (second derivatives are not required) is at: www.mathcentre.ac.uk/resources/uploaded/mc-ty-parametric-2009-1.pdf **Extension activity:** Learners investigate interesting curves expressed in parametric form using a graph plotter. There are many websites with good examples, for instance this one gives a selection of equations: https://cims.nyu.edu/~kiryl/Calculus/Section_9.1--Parametric_Curves/Parametric_Curves.pdf For implicit differentiation, start with the definition of implicit and explicit functions. Ask learners to consider e.g. $y^2 = x$, rewrite it as $y = x^{\frac{1}{2}}$ then differentiate to obtain $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x^{-\frac{1}{2}}$. They can rewrite this as $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2y}$ leading to the statement $2y\frac{\mathrm{d}y}{\mathrm{d}x} = 1$. Repeat this exercise, as a whole class or in groups, with several similar examples (powers of *y*) so that learners identify a pattern. Show learners terms of various types: they now know how to differentiate powers of *x* or *y* with respect to *x*. Introduce the idea of a product term by asking them to differentiate equations such as $xy = x$ and $x^2y^3 = 4$ implicitly using the product rule and by rearranging them and differentiating *y* with respect to *x*. Now ask learners to work through an equation from left to right and differentiate it implicitly without rearranging it first. (Give them equations which cannot be rearranged to prevent them doing this.) Useful examples or worksheets are at: www.khanacademy.org/math/differential-calculus/taking-derivatives/implicit_differentiation/v/implicit-derivative-of-x-y-2-x-y-1 www.intmath.com/differentiation/8-derivative-implicit-function.php

Subject content	Suggested teaching activities
	http://cdn.kutasoftware.com/Worksheets/Calc/03%20-%20Implicit%20Differentiation.pdf

Past and specimen papers
Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support **(F)** 9709 Mathematics 9709 2020 Specimen Paper 3, Question 5(a)

3.5 Integration

Learning objectives	Suggested teaching activities
• extend the idea of 'reverse differentiation' to include the integration of e^{ax+b}, $\frac{1}{ax+b}$, $\sin(ax+b)$, $\cos(ax+b)$, $\sec^2(ax+b)$ and $\frac{1}{x^2+a^2}$; including examples such as $\frac{1}{2+3x^2}$	Start with a quick review of integration from Pure Mathematics 1.8 'Integration', perhaps as a question and answer session with learners writing on mini whiteboards and holding up their responses. This will enable you to assess all learners' understanding before moving on to examples in this section. Divide learners into groups and give them sets of expressions to integrate. Ask them to consider what would need to be differentiated to obtain the given expression, then to work out some general principles. A good approach for integration involving logarithmic functions is at: www.mathcentre.ac.uk/resources/uploaded/mc-ty-inttologs-2009-1.pdf **(I)** Some of the examples may be beyond the range of this syllabus. Textbooks will include exercises on integrating all of these types of function, including finding areas. **(I)** An interesting task looking at finding the area between $\sin x$ and $\sin(2x)$ is at: https://undergroundmathematics.org. Select 'Calculus of trigonometry & logarithms' on the map and look for the Review question 'Can we find the area between $\sin x$ and $\sin 2x$?' **(I)**
• use trigonometrical relationships in carrying out integration, e.g. use of double-angle formulae to integrate $\sin^2 x$ or $\cos^2(2x)$	Ask learners to recall the three forms of the trigonometric identity for $\cos 2x$ and then to use them to rewrite $\cos^2 x$ and $\sin^2 x$ in terms of $\cos 2x$. Introduce learners to integrals of the type $\int 2\sin x\cos x\,dx$, $\int \cos^2 2x\,dx$ and $\int \tan^2 3x + 1\,dx$. Appropriate textbooks will have examples of these. Try to relate them to areas and also to simple first order differential equations, for example: find the equation of the curve, with gradient function $\frac{dy}{dx} = 2\sec^2 x + 1$ for $0 \le x < \frac{\pi}{2}$, which passes through the point $x = \frac{\pi}{4}$ **(I)**
• integrate rational functions by	You could cover this section with the section on partial fractions (see 1. Algebra) or later, perhaps checking learners'

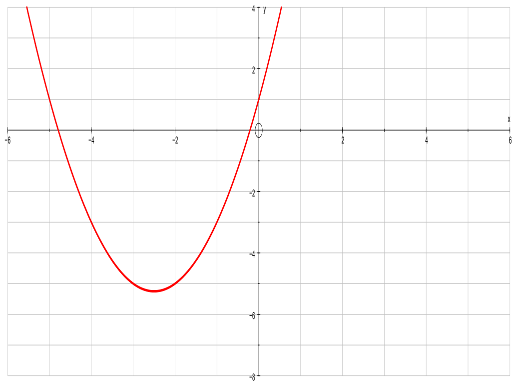
Learning objectives	Suggested teaching activities
means of decomposition into partial fractions; restricted to types of partial fractions as specified in topic 3.1 above	understanding by setting them some preparatory questions involving linear denominators. **(I)** By considering the integration of $\int \frac{a}{ax+b}\,\mathrm{d}x$ to $\ln(ax+b)+c$, you can show learners the link to the next section which deals with non-linear denominators. Learners will need to practise definite integrals of this type, using laws of logarithms to simplify their answers when appropriate. (You will need to cover the laws from section 2. Logarithms and exponentials first). Textbooks will have many suitable questions for learners to practise.
• recognise an integrand of the form $\frac{kf'(x)}{f(x)}$ and integrate such functions, e.g. integration of $\frac{x}{x^2+1}$, tan x	As a whole class exercise, you can extend the work done in the previous section by considering different examples of the form $\frac{kf'(x)}{f(x)}$ where f(x) is non-linear. You will find some useful examples, some of which relate to physical situations, at this link: www.intmath.com/methods-integration/2-integration-logarithmic-form.php Ask learners to work in pairs or individually to integrate functions such as tan x, cot x, tan kx and cot kx, checking their answers by differentiating them. Integration of this type is often needed when finding the solutions of first order differential equations, so you could give learners more practice at this type of integration later (see section 8. Differential Equations.)
• recognise when an integrand can usefully be regarded as a product, and use integration by parts, e.g. integration of x sin 2x, x^2e^{-x}, ln x, $x\tan^{-1}x$	**Extension activity:** Challenge more able learners to start with the product rule and see if they can derive a formula for integrating a product. They may need some hints to rearrange the product rule then integrate all the terms with respect to x. A resource that includes the derivation as well as reasons for using the formula, and a set of questions is at: www.mathcentre.ac.uk/resources/uploaded/mc-ty-parts-2009-1.pdf As a group or individual activity, ask learners to think how they could integrate $\ln x$ or $\ln(2x+1)$. See if they can deduce that they can integrate it by parts if they form a product to start with.

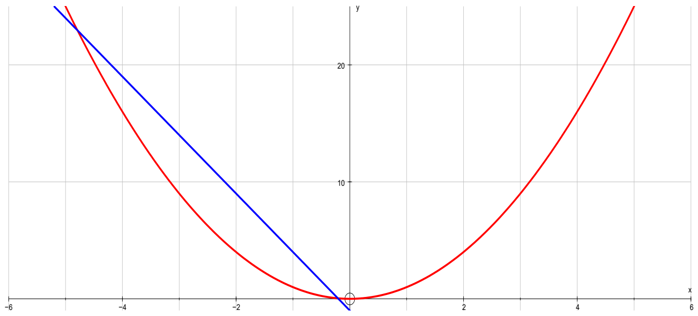
Learning objectives	Suggested teaching activities
	An activity which introduces integrating ln$x$ is at: https://undergroundmathematics.org. Select 'Calculus of trigonometry and logarithms' on the map and look for the task 'Inverse integrals'. Some helpful video resources are available at: www.khanacademy.org/math. Search for 'Integration by parts'. These might be helpful for a flipped learning approach to the topic. **(I)** Worksheets to practise the technique are at: www.tes.com/resources/search. Search for 'Integration by parts' and look for 'Maths KS5 Core 4: Integration by parts worksheet' by chuckieirish or 'Integration by parts' by phildb. **(I)** Other areas of the syllabus will need integration by parts, e.g. first order differential equations, so learners will need to recognise 'products' when attempting questions. Suitable past/specimen papers for practice and/or formative assessment include **(I)(F)**: 2020 Specimen Paper 3, Q5b;
• use a given substitution to simplify and evaluate either a definite or an indefinite integral, e.g. to integrate $\sin^2 2x \cos x$ using the substitution $u = \sin x$	Start with a simple example which can easily be checked by other means, e.g. $\int (2x+5)^3 \, dx$ by using the substitution $u = 2x+5$ or by expanding first. This will give learners confidence that the right substitution will work. Textbooks contain many useful examples of both indefinite and definite integrals. There are also examples at: www.mathcentre.ac.uk/resources/workbooks/mathcentre/web-integrationbysub-tony.pdf For definite integrals, explain to learners that they will also need to convert their limits to fit the new variable. This saves them having to substitute back the original variable and may reduce the risk of errors. An example is shown here: /wwwf.imperial.ac.uk/metric/metric_public/integration/substitution/substitution.html Using a graph plotter, help learners to visualise the substitution as transformation of the area under a graph. Plot both the original function and the new function after substitution, then compare the corresponding areas under the two graphs between the limits for each function. A worksheet that learners can use for practice and consolidation is at: www.tes.co.uk/teaching-resource/integration-by-substitution-worksheet-6152845 **(I)** A variety of resources that are helpful to recap or summarise the work on differentiation and integration are at: www.tes.com/resources/search. Search for:

Learning objectives	Suggested teaching activities
	• 'A level maths: Spotting the integration worksheet' and look for the resource by phildb **(I)** • 'A level maths: Integration revision sheets' and look for the resource by phildb (summary of techniques) **(I)** • 'Integration by parts worksheet' and look for the resource by SRWhitehouse for a set of questions focussed mainly on integration by parts, but with some examples of using substitutions **(I)** • 'A2 Differentiation and Integration' and look for the resource by thatsmyboy for a summary of differentiation and integration techniques. **(I)**

Past and specimen papers
Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support **(F)** 9709 Mathematics 2020 Specimen Paper 3, question 5(b)

3.6 Numerical solution of equations

Learning objectives	Suggested teaching activities
• locate approximately a root of an equation, by means of graphical considerations and/or searching for a sign change, e.g. finding a pair of consecutive integers between which a root lies.	Introduce this topic by using a graph plotter to demonstrate both sign changes and graphical considerations e.g. $y = x^2 + 5x + 1$: Change of sign  You can see clearly that there are solutions to the equation $x^2 + 5x + 1 = 0$ in the intervals $-5 < x < -4$ and $-1 < x < 0$. Learners consider the sign of *y* either side of the points of intersection of the curve with the *x*-axis i.e. using the boundaries above. Demonstrate also that the same result may be obtained by plotting $y = x^2$ against $y = -5x - 1$:

Learning objectives	Suggested teaching activities
	 Learners will need to practise examples of both types. Encourage them to set out their work clearly and accurately. For example, to show that the equation $x^2 = -5x - 1$ has a solution in the interval $-5 < x < -4$, learners should state 'Let $f(x) = x^2 + 5x + 1$' then write the equation as $f(x) = 0$. By calculating and writing down the values of $f(-5)$ and $f(-4)$, they can demonstrate that there is a sign change and state their conclusion e.g. ' There is a change of sign, so a solution lies in the interval $-5 < x < -4$'. A useful overview of the topic, with examples is at: www.cimt.org.uk/projects/mepres/alevel/pure_ch19.pdf
• understand the idea of, and use the notation for, a sequence of approximations which converges to a root of an equation	The second part of this chapter deals with convergence to a root of an equation: www.cimt.org.uk/projects/mepres/alevel/pure_ch19.pdf **Extension activity:** The first part of this chapter demonstrates a formal approach to the idea of a sequence of approximations converging to a root of an equation: www-solar.mcs.st-andrews.ac.uk/~clare/Lectures/num-analysis/Numan_chap2.pdf You could use it with able learners or perhaps with a whole class. It explains how an iterative formula generates the sequence; this is the next learning objective.
• understand how a given simple iterative formula of the form $x_{n+1} = F(x_n)$ relates to the equation being solved, and use a given iteration, or an iteration based on a	A video tutorial which learners could watch independently or as a whole class is at: www.tes.com/teaching-resource/iteration-6201516 Iterative formulae are covered in this chapter, which includes examples and activities for learners to try: www.cimt.org.uk/projects/mepres/alevel/pure_ch19.pdf

Learning objectives	Suggested teaching activities
given rearrangement of an equation, to determine a root to a prescribed degree of accuracy; knowledge of the condition for convergence is not included, but an understanding that an iteration may fail to converge is expected	It is a good idea for learners to make full use of their calculator for the iteration process. Using the ANS (answer) key will save them time in finding a root of an equation. For example: Using the iterative formula $x_{n+1} = 3 - \frac{1}{x_n}$ with $x_0 = 3$, show successive iterations to five decimal places and a final answer to three decimal places. • Start by entering the value of x_0 into the calculator: press '3' then '=' (or 'enter', depending on the calculator), so 3 appears as an answer. • Key in the right hand side of the iterative formula, replacing x_n with ANS (or the key that displays a previous answer) i.e. $3-(1 \div \text{ANS})$. The calculator will display 2.666666667. Write this down to five decimal places. • Keep pressing the '=' key and successive iterations will appear. Write down as many as the question requires, all correct to 5 decimal places: 2.62500 2.61905 2.61818 2.61806 2.61804 2.61803 • You have now done enough iterations to show that an answer of 2.618 is correct to three decimal places. Learners will need practice at entering the correct formula into their calculator, using brackets where necessary.

3.7 Vectors

Learning objectives	Suggested teaching activities
• use standard notations for vectors, i.e. $\begin{pmatrix} x \\ y \end{pmatrix}$, $x\mathbf{i}+y\mathbf{j}$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, $\overrightarrow{AB}$, **a**	Start by asking learners to give examples of vector and scalar quantities, with explanations. Introduce the notation for the coordinate axes, first in two dimensions then extending to three dimensions, and then follow on with the notation for vectors. Encourage learners to use correct vector notation when working through problems. A quick practice exercise is at: www.bbc.co.uk/schools/gcsebitesize/maths/geometry/vectorshirev1.shtml Learners may find it easier to use vectors in column vector form rather than component form. Also introduce the magnitude of a vector quantity at this point.
• carry out addition and subtraction of vectors and multiplication of a vector by a scalar, and interpret these operations in geometrical terms, e.g. '*OABC* is a parallelogram' is equivalent to $\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{OC}$; the general form of the ratio theorem is not included, but understanding that the mid-point of *AB* has position vector $\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OB}\right)$ is expected	Many learners will already be familiar with adding and subtracting vectors, so a revision activity is a good starting point. This would also help those meeting vectors for the first time. A useful set of diagrams explaining vector notation, addition and subtraction is at: www.mathsisfun.com. Search for 'Vectors'. Encourage learners to study this independently and come to the lesson ready to start solving problems. **(I)** Examples of geometrical problems involving addition and subtraction are at: www.bbc.co.uk/schools/gcsebitesize/maths/geometry/vectorshirev2.shtml A problem involving addition and subtraction is at: https://nrich.maths.org. Search for 'Vector Walk'. A problem-solving task which links nicely to this area of the topic is at: https://undergroundmathematics.org. Select 'Vector geometry' on the map. Look for the task 'Hit the spot' which looks at addition of vectors using different notation. Some useful resources that can be used for this topic area are at: www.tes.com/resources/search. Search for: • 'Maths vectors starter plenary powerpoint' by tristanjones for a PowerPoint of questions of increasing difficulty using vector addition, subtraction and multiplication by a scalar in geometrical terms. This is designed to be used with interactive voting systems, but could be adapted by asking learners to vote in other ways. **(F)** • 'Introduction to vectors' by SRWhitehouse is a collection of resources that would be useful for the whole of the vectors section of Pure Mathematics 3. **(I)(F)**

Learning objectives	Suggested teaching activities
• calculate the magnitude of a vector, and use unit vectors, displacement vectors and position vectors, in 2 or 3 dimensions	The magnitude of a vector needs to be dealt with fairly early on in the topic. Many learners may have come across it already. Start with vectors such as $\begin{pmatrix}3\\4\end{pmatrix}$, $\begin{pmatrix}5\\-12\end{pmatrix}$, and ask learners to work out the magnitude, using a sketch if necessary. Go on to other vectors e.g. $\begin{pmatrix}4\\-5\end{pmatrix}$ and ensure that the learners give the magnitude in surd form, $\sqrt{41}$ in this case, rather than as a rounded decimal. Exact forms will often be required as answers to problems. Many learners have difficulties with the concept of a unit vector. Start by drawing some vectors on graph paper or a board, e.g. $\begin{pmatrix}3\\4\end{pmatrix}$, $\begin{pmatrix}5\\-12\end{pmatrix}$. Show how the first vector can be 'split' into five separate vectors each of length one unit. Verify that the same principle applies to the second vector. Ask the learners to deduce how to find a unit vector. Extend this work to involve 3-dimensional vectors. Define a displacement vector as follows: The **displacement** of an object is defined as the **vector distance** from an initial point to a final point. It is important for learners to understand that this is different from the **distance travelled**. Also point out to learners that direction is crucial, e.g. displacement vector $\begin{pmatrix}2\\-5\end{pmatrix}$ is different from displacement vector $\begin{pmatrix}-2\\5\end{pmatrix}$, although the distance travelled is the same in both cases. Learners need to be aware of the unique nature of position vectors, e.g. a point *A* has position vector $\begin{pmatrix}2\\-1\\4\end{pmatrix}$ relative to an origin *O* so there is only one possible position for point *A*. Some resources that can be used for further practice using vectors are at: www.tes.com/resources/search. Search for: • 'Vectors worksheet Higher GCSE' and look for the resource by judsonb. This provides some good examples.

Learning objectives	Suggested teaching activities
	Learners could work in groups on separate questions and present their answers to the class as a whole to check their method and use of notation. **(I)** • 'Ericas errors vectors' (without apostrophe) and look for 'Erica's errors on vectors' and 'Erica's errors on vectors 2' by alutwyche for two pieces of work which learners should look for errors in and correct. **(I)(F)** • 'Tarsia -vectors' for a variety of different Tarsia puzzles. The resource 'Tarsia – vectors for A level 1' by MrBartonMaths is suitable for this section of the syllabus, but other card sorts are available that could be used with other sections of the vectors topic. **(I)(F)**
• understand the significance of all the symbols used when the equation of a straight line is expressed in the form **r** = **a** + *t***b**, and find the equation of a line, given sufficient information e.g. finding the equation of a line given the position vector of a point on the line and a direction vector, or the position vectors of two points on the line	Start by asking learners to use position vectors to find the vector equation of a straight line if the line passes through a point with position vector **a** and is parallel to a vector **b**. This will give them the idea of jumping from the origin to the line, then moving along it. Ask learners what it means to choose different values of the scalar *t* and reinforce the concept of the line as a set of points, each of which is described in the form: **r** = (position vector of a point on the line) + *t*(direction vector of the line) Alternatively, a task which can be used to introduce the vector equation of a line is at: https://undergroundmathematics.org. Select 'Vector geometry' on the map and look for the task 'Vector squares'. **(I)** Working in three dimensions may help learners to see why they need a vector equation for a line; *y* = *mx* + *c* is not enough and vectors are a powerful tool. Useful introductory examples are at: wwwf.imperial.ac.uk/metric/metric_public/vectors/vector_coordinate_geometry/vector_equation_of_line.html www.cimt.org.uk/projects/mepres/alevel/fpure_ch5.pdf The second link will also be useful in the following sections. Learners can practise using this form of the equation. The link below leads to three files; the file 'Vector equation of a line' provides examples of this type in two and three dimensions (log in for free download): www.tes.co.uk/teaching-resource/vector-equation-of-a-line-6146907 **(I)**
• determine whether two lines are parallel, intersect or are skew, and find the point of intersection of two lines when it exists; calculation of the shortest distance between two skew lines is not required; finding the equation of the common perpendicular to two skew lines is also not required	From the vector equation of the line, ask learners how they could determine whether lines are parallel. Try giving some examples of vector equations in different forms, e.g. the line $\mathbf{r}=\begin{pmatrix}2\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}5\\6\\7\end{pmatrix}$ is parallel to the line $\mathbf{r}=\begin{pmatrix}-3+10\mu\\12\mu\\1+14\mu\end{pmatrix}$ this is easy to see if learners rewrite the second one in the same form. For intersecting lines, there is some value of λ and μ that satisfies all three equations for the vector components *x*, *y*

Learning objectives	Suggested teaching activities
	and *z*. There is an example at: www.cimt.org.uk/projects/mepres/alevel/fpure_ch5.pdf. Activity 2 may be useful; learners need to decide which pairs of lines intersect. Alternatively, a task which could be used to introduce this topic is at: https://undergroundmathematics.org. Select 'Vector geometry' on the map. The task 'Three lines' gets learners to think about whether lines are parallel, perpendicular, intersect etc. **(I)** Learners often find skew lines difficult to visualise, so show them an image (Google 'skew lines' or illustrate the geometry by holding up two long rulers). From one direction, the rulers look as though they are intersecting in a plane, but from a perpendicular direction they are clearly not. This will tie in with solving equations: you can find values for λ and μ from two of the equations (a plane) but the values do not fit the third equation (3rd dimension). A good activity for building learner fluency with identifying parallel lines, perpendicular lines and other properties of lines in 2D given in vector form is at: https://undergroundmathematics.org. Select 'Vector geometry' on the map before looking for the task 'Lots of vector lines!'. This website also has a variety of other activities and review questions relevant to vectors. **(I)(F)**
• use formulae to calculate the scalar product of two vectors, and use scalar products in problems involving lines and points; e.g. finding the angle between two lines, and finding the foot of the perpendicular from a point to a line; questions may involve 3D objects such as cuboids, tetrahedra (pyramids), etc.; knowledge of the vector product is not required	Introduce the scalar product of two vectors with a formal definition. Ask learners to work out the angle between two vectors in two dimensions, e.g. $\begin{pmatrix}5\\2\end{pmatrix}$ and $\begin{pmatrix}3\\12\end{pmatrix}$, using trigonometry and hence to work out the scalar product $\left\|\begin{pmatrix}5\\2\end{pmatrix}\right\|\left\|\begin{pmatrix}3\\12\end{pmatrix}\right\|\cos\theta$, where θ is the angle they have just found. Ask them if they can deduce a quicker process of getting the scalar product. Follow with a formal proof (for two or three dimensions, or both) after learners have worked out the scalar products of pairs of unit vectors such as $\mathbf{i}$ and $\mathbf{j}$, $\mathbf{i}$ and $\mathbf{k}$. A good overview of the scalar product. is at: www.mathsisfun.com . Search for 'Dot product'. Show learners that the formula for the scalar product may be rewritten as $\cos\theta = \dfrac{\mathbf{a}.\mathbf{b}}{\|\mathbf{a}\|.\|\mathbf{b}\|}$ and use it to find angles in triangles and parallelograms. Learners should be able to state the requirement for perpendicularity without much trouble.

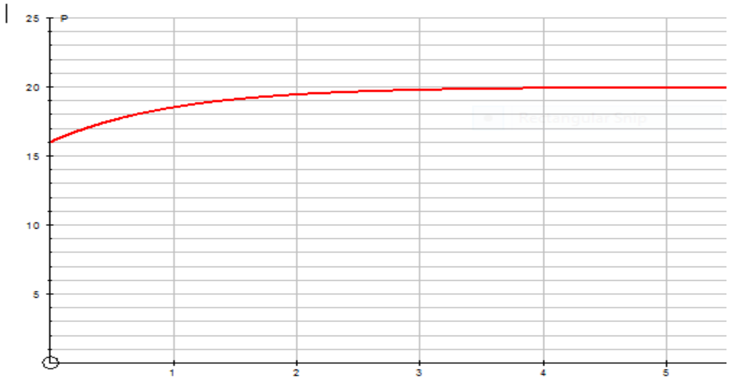
Learning objectives	Suggested teaching activities
	Most textbooks will have plenty of examples for practice. A worksheet with solutions (log in for free download) is at: www.tes.co.uk/teaching-resource/c4-maths-vectors-worksheet-6096103 **(I)(F)** To find the coordinates of the point of intersection, learners just need to substitute their value of λ or μ to find the position vector and hence coordinates of the point. Further examples and questions are at: www.cimt.org.uk/projects/mepres/alevel/fpure_ch5.pdf. **(I)** A range of worksheets with further examples and questions is at: www.tes.com/resources/search/. Search for: • 'A level vectors worksheet', look for the resource by SRWhitehouse • 'Vector equation of a line', look for the resource by SRWhitehouse • 'C4 Maths Vectors Worksheet', look for the resource by chuckieirish • 'Questions on vectors', look for the resource by SRWhitehouse **(I)**

Past and specimen papers
Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support **(F)** 9709 Mathematics 2020 Specimen Paper 3, question 8 (a)

3.8 Differential equations

Learning objectives	Suggested teaching activities
• formulate a simple statement involving a rate of change as a differential equation; the introduction and evaluation of a constant of proportionality, where necessary, is included	Start by asking learners to think about the gradient function $\frac{\mathrm{d}y}{\mathrm{d}x}$, as the rate of change of *y* with respect to *x*. Using other variables, introduce them to other rates of change such as $\frac{\mathrm{d}s}{\mathrm{d}t}$, the rate of change of variable *s* (which could represent distance) with respect to a variable *t* (which could represent time). An introduction to forming differential equations and solving first order differential equations, including some real-life examples is at: www.slideshare.net/davidmiles100/core-4-differential-equations-1 Three files, one of which is a worksheet on forming and solving differential equations (log in for free download) are at: www.tes.co.uk/teaching-resource/ks5-core-4-c4-first-order-differential-equations-6095650 An interesting matching game that requires learners to match first order differential equations and descriptions is at: www.tes.co.uk/teaching-resource/matching-differential-equations-to-descriptions-6242133 You could use it to check learners' understanding.
• find by integration a general form of solution for a first order differential equation in which the variables are separable; including any of the integration techniques from topic 3.5 above	Start with a simple differential equation e.g. $\frac{\mathrm{d}y}{\mathrm{d}x}=x^2$ and ask learners to find a solution to this. Introduce the term 'general solution' and emphasise the importance of the constant of integration. By considering $\frac{\mathrm{d}y}{\mathrm{d}x}=y^2$, lead on to the idea of separating variables. Learners will benefit from practice at separating variables, so give them a good variety of questions on this before they move on to solving equations of greater complexity. Use this section to help learners revisit all types of integration and to practise simplifying logarithms using the laws. They also need to be aware that terms such as e^{x+c}, where *c* is a constant, may be written in the form $A\mathrm{e}^{x}$, and to practise rewriting solutions in the form required for each question.

Learning objectives	Suggested teaching activities
	Three files, one of which contains notes and examples on general solutions of first order differential equations (log in for free download) are at: www.tes.co.uk/teaching-resource/ks5-core-4-c4-first-order-differential-equations-6095650 Practice for both general and particular solutions is at: http://mselwardclass.pbworks.com/w/file/fetch/72617471/Solve%20First%20Order%20DiffEQ.pdf **(I)**
• use an initial condition to find a particular solution	Introduce the idea that a particular solution relates to specific conditions given in the question, and that the conditions lead to finding a value for the constant. Learners will consolidate their work on general solutions when working through problems requiring particular solutions. Three files, one of which contains notes and examples on particular solutions of first order differential equations (log in for free download) are at: www.tes.co.uk/teaching-resource/ks5-core-4-c4-first-order-differential-equations-6095650 Practice for both the general and particular solutions is at: http://mselwardclass.pbworks.com/w/file/fetch/72617471/Solve%20First%20Order%20DiffEQ.pdf **(I)** An interactive exercise on particular solutions of first order differential equations is at: http://worksheets.tutorvista.com/differential-equations-worksheet.html **(I)** Examples and practice questions on finding particular solutions, as well as interesting activities based on a population model, that would help learners to understand that a model might break down, are at: www.cimt.org.uk/projects/mepres/alevel/pure_ch18.pdf
• interpret the solution of a differential equation in the context of a problem being modelled by the equation; where a differential equation is used to model a 'real-life' situation, no specialised knowledge of the context will be required	Having solved a differential equation, learners often need to interpret their solution in context. Sometimes a graph can help them to deduce what is happening.

Learning objectives	Suggested teaching activities
	 The graph above shows the particular solution to the equation $\frac{\mathrm{d}P}{\mathrm{d}t} = -(P-20)$ where *P* represents the size of a population, in 1000s, and *t* represents time in years. It is given initially that when $t = 0$, $P = 16$. This leads to the particular solution $P = 20 - 4\mathrm{e}^{-t}$. You can see from the graph that, as *t* increases, $P \to 20$, so you can conclude that, over time, the population increases and approaches 20 000 but never reaches it. Alternatively, you can see this algebraically from the particular solution: as *t* increases, $e^{-t} \to 0$ so $P \to 20$. Learners will benefit from practising exam type questions to build confidence in this type of interpretation. Textbooks will also provide practice questions for learners to work through. **(I)**

3.9 Complex numbers

Learning objectives	Suggested teaching activities
• understand the idea of a complex number, recall the meaning of the terms real part, imaginary part, modulus, argument, conjugate, and use the fact that two complex numbers are equal if and only if both real and imaginary parts are equal; notations **Re***z*, **Im***z*, |*z*|, **arg***z*, *z** should be known; the argument of a complex number will usually refer to an angle θ such that $-\pi < \theta \le \pi$, but in some cases the interval $0 \le \theta < 2\pi$ may be more convenient; answers may use either interval unless the question specifies otherwise	Introduce the concept of complex numbers by asking learners to solve an equation e.g. $x^2 + 6x + 25 = 0$. Using their knowledge of the discriminant, or by using the quadratic formula, learners can deduce that the equation has no real roots. Ask them to write down the two square roots of –64 using $\mathrm{i} = \sqrt{-1}$. They can give the solutions to the quadratic equation as $x = -3 + 4\mathrm{i}$ and $x = -3 - 4\mathrm{i}$. Introduce the term ‘complex number’ for these numbers with a real part and an imaginary part. You may wish to mention that engineers generally use j rather than i for $\sqrt{-1}$. Plotting complex numbers on an Argand diagram will help learners to visualise them as two-dimensional numbers. Use the diagram to introduce the terms ‘conjugate’, ‘modulus’ and ‘argument’ together with the appropriate notation and conventions for these. Learners then practise plotting basic examples; you will find many in textbooks. The article here demonstrates a similar approach and includes some investigations which learners may find interesting: http://nrich.maths.org/1403 Learners can move on to consider problems which require them to equate real and imaginary parts. Draw an analogy with the process of equating coefficients. For example: ‘Given that the complex numbers $(a+1)+2\mathrm{i}$ and $2a+(3a-b)\mathrm{i}$ are equal, find the values of *a* and *b*’. Four files: a summary of the above points, examples and a matching activity which you could use in groups to check learners’ understanding (log in for free download) are at: www.tes.co.uk/teaching-resource/complex-numbers-6147229
• carry out operations of addition, subtraction, multiplication and division of two complex numbers expressed in Cartesian form x + iy; for calculations involving multiplication or division, full details of the working should be shown	All of these methods will be familiar to learners from other areas of mathematics: • addition and subtraction of two complex numbers is similar to adding and subtracting vectors; learners will find it useful to deal with this both algebraically and using an Argand diagram • multiplication of two complex numbers is similar to expanding brackets • division of one complex number by another is similar to rationalising surds in the denominator. Although the geometrical interpretation of these operations using an Argand diagram appears in a later section, you may wish to cover it here along with the algebra. Discuss with learners examples of each type then set them plenty of practice. **(I)**

Learning objectives	Suggested teaching activities
	Several worksheets on the topics in this section to provide learners with practice and consolidation are at: www.mathworksheetsgo.com/sheets/algebra-2/complex-numbers/imaginary-numbers-worksheet.php **(I)** There are also interesting investigations and spot-the-error exercises.
• use the result that, for a polynomial equation with real coefficients, any non-real roots occur in conjugate pairs, e.g. in solving a cubic or quartic equation where one complex root is given • represent complex numbers geometrically by means of an Argand diagram	Start by giving learners some basic equations to solve, e.g. $z^2 - 14z + 53 = 0$ $2z^3 - 4z^2 - 5z - 3 = 0$. They may need a hint to find one root using the factor theorem ($z = 3$). Give learners more examples of this type and ask them to make a deduction from their results. You can then give them a more advanced example e.g. 'Given that $2-3\mathrm{i}$ is one of the roots of the equation $z^4 + 2z^3 - z^2 + 38z + 130 = 0$, solve the equation completely'. Appropriate textbooks will have plenty of questions for learners to practise. You may already have introduced the Argand diagram in the previous section. This link will help learners to think about geometrical relationships between points on the Argand diagram: http://nrich.maths.org/9859/note
• carry out operations of multiplication and division of two complex numbers expressed in polar form $r(\cos\theta + \mathrm{i}\sin\theta) \equiv re^{i\theta}$; including the results $\|z_1 z_2\| = \|z_1\|\|z_2\|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$, and corresponding results for division	This chapter provides useful examples and exercises on the polar form of complex numbers, making use of the Argand diagram: www.cimt.org.uk/projects/mepres/alevel/fpure_ch3.pdf When learners are competent with the polar form, set them examples of multiplication and division and ask them to deduce what happens to the moduli and arguments. They could interpret their results using the Argand diagram too. Interactive questions for learners to answer and assess their progress are at: www.khanacademy.org/math/precalculus/imaginary_complex_precalc/exponential-form-complex-numbers/e/multiplying_and_dividing_complex_number_polar_forms You will find additional examples in appropriate textbooks.
• find the two square roots of a complex number e.g. the square roots of 5 + 12i in exact Cartesian form; full details of the working should be shown	Start by having a class discussion with learners: how could they find the square roots of a complex number such as $3+4\mathrm{i}$? With careful questioning, encourage them to write it in the form $(a+b\mathrm{i})^2 = 3+4\mathrm{i}$, where $a+b\mathrm{i}$ is a square root of $3+4\mathrm{i}$, and to equate real and imaginary parts. This approach is demonstrated at: www.examsolutions.net/tutorials/square-roots-complex-number Textbooks will have examples for learners to practise.

Learning objectives	Suggested teaching activities
• understand in simple terms the geometrical effects of conjugating a complex number and of adding, subtracting, multiplying and dividing two complex numbers	You may already have covered this in earlier sections if you used Argand diagrams as well as algebra. There is a useful interactive resource here for visualising multiplication and division on an Argand diagram: www.furthermaths.org.uk/files/sample/files/ComplexMultiplication.html. Either use it as a demonstration for the whole class or individual learners can use it to predict their answers and check them.
• illustrate simple equations and inequalities involving complex numbers by means of loci in an Argand diagram, e.g. $\|z - a\| < k$, $\|z-a\| = \|z-b\|$, $\arg(z - a) = a$	Start by asking learners what they understand by the words 'locus' and 'loci' and ask them to suggest any examples. One example is the circle: the locus of a point which moves such that it is always a constant distance from a fixed point. Teaching points and ideas are at: www.ncetm.org.uk/self-evaluation/browse/topic/674 Using an example e.g. $\|z-(3+4\mathrm{i})\| = 5$, ask learners to plot the complex number $3+4\mathrm{i}$ then to consider the significance of the 5, the z and the modulus signs. Draw the parallel with the definition of a circle above and plot the circle described by this equation. Extend this reasoning to the inequality $\|z-(3+4\mathrm{i})\| < 5$, asking learners to shade the appropriate region on their diagram. Using an example such as $z+3-2\mathrm{i}$ encourage learners to rewrite it as $z-(-3+2\mathrm{i})$ then to plot a circular locus based on the point $-3+2\mathrm{i}$. You could also demonstrate to learners that they can find the Cartesian equations of loci by writing z as $x+\mathrm{i}y$. The Cartesian form could be useful for verifying that, for examples of the type $\|z-a\| = \|z-b\|$, the locus is a perpendicular bisector. For examples of the type $\arg(z - a) = a$, it is important that learners realise only half lines are needed. Examples and exercises that you could either use in class or learners could use independently for revision are at: www.ilovemaths.com/3argandplane.asp A worksheet on loci in the complex plane that could be used for practice or consolidation is at: www.tes.co.uk/teaching-resource/loci-in-the-complex-plane-6152307 **(I)**

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