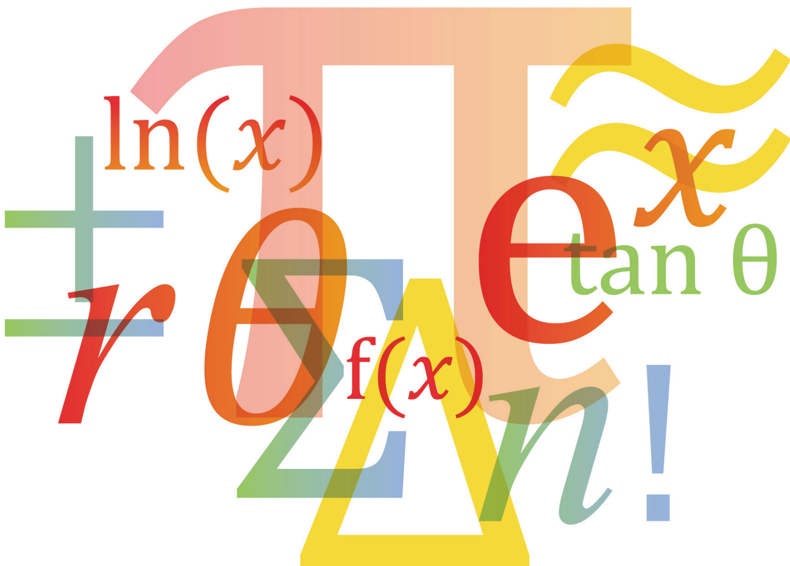
Cambridge Assessment International Education

Example Candidate Responses – Paper 3

Cambridge International AS & A Level Mathematics 9709

For examination from 2020



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Introduction

The main aim of this booklet is to exemplify standards for those teaching Cambridge International AS & A Level Mathematics 9709 and to show how different levels of candidates' performance (high, middle and low) relate to the syllabus requirements.

In this booklet, candidate responses have been chosen from the November 2020 exam series to exemplify a range of answers.

For each question, the response is annotated with a clear explanation of where and why marks were awarded or omitted. This is followed by examiner comments on how the answer could have been improved. In this way, it is possible for you to understand what candidates have done to gain their marks and what they could do to improve their answers. There is also a list of common mistakes candidates made in their answers for each question.

This document provides illustrative examples of candidate work with examiner commentary. These help teachers to assess the standard required to achieve marks beyond the guidance of the mark scheme. Therefore, in some circumstances, such as where exact answers are required, there will not be much comment.

The questions and mark schemes used here are available to download from the School Support Hub. These files are:

November 2020 Question Paper 32

November 2020 Paper 32 Mark Scheme

Past exam resources and other teaching and learning resources are available on the School Support Hub:

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How to use this booklet

Example Candidate Response – middle	Examiner comments
1 Solve the equation $$\ln(1 + e^{-3x}) = 2.$$ Give the answer correct to 3 decimal places. [3] $1 + e^{-3x} = e^2$ 1 ~~$e^2 - \frac{1}{e^{3x}} = 1 = 0$~~ ~~2~~ ~~$\ln 1 + \ln e$~~ $\ln 1 - 3x = 2$ 2 $0 - \frac{3x}{-3} = \frac{2}{-3}$ $x = 0.667$ $x = -0.667$	1 The candidate takes the correct first step, removing the logarithm. 2 The candidate uses an incorrect formula $\ln(A + B) = \ln A + \ln B$ so no further marks are awarded. **Total mark awarded = 1 out of 3**

Answers are by real candidates in exam conditions. These show you the types of answers for each level. Discuss and analyse the answers with your learners in the classroom to improve their skills.

Examiner comments are alongside the answers. These explain where and why marks were awarded. This helps you to interpret the standard of Cambridge exams so you can help your learners to refine their exam technique.

How the candidate could have improved their answer

The candidate could have paused to think about the basic rules for this topic. Us
no further marks could be awarded.

This section explains how the candidate could have improved each answer. This helps you to interpret the standard of Cambridge exams and helps your learners to refine their exam technique.

Common mistakes candidates made in this question

- Incorrect removal of brackets to give $\ln 1 + (-3x) = 2$.
- A correct first step followed by incorrect use of logarithms to give $\ln 1 + (-3x) = 2$.
- Calculator errors.
- Not giving the final answer to the required accuracy.

Often candidates were not awarded marks because they misread or misinterpreted the questions.

Lists the common mistakes candidates made in answering each question. This will help your learners to avoid these mistakes and give them the best chance of achieving the available marks.

Question 1

Example Candidate Response – high

1 Solve the equation

$$\ln(1 + e^{-3x}) = 2.$$

Give the answer correct to 3 decimal places. [3]

$\ln(1 + e^{-3x}) = 2 \times \ln e$

$\cancel{\ln}(1 + e^{-3x}) = \cancel{\ln} e^2$

$1 + e^{-3x} = e^2$ (1)

$e^{-3x} = e^2 - 1$

$e^{-3x} = 6{\cdot}389$

$\ln e^{-3x} = \ln 6{\cdot}389$ (2)

$-3x \ln e = 1{\cdot}305$

$-3x = 1{\cdot}305$

$x = -\dfrac{1{\cdot}305}{3}$ (3)

$x = -0{\cdot}435$

Examiner comments

(1) The first step is correct.

(2) This line is correct. On the next line, the left-hand side is simplified correctly, but there is a calculation error on the right-hand side (the correct value should be 1.854...).

(3) The candidate uses a correct method but makes an accuracy error.

Total mark awarded = 2 out of 3

How the candidate could have improved their answer

The method used was correct. The candidate could have used any time available at the end of the exam to check the accuracy of their calculations.

Example Candidate Response – middle

1 Solve the equation

$$\ln(1 + e^{-3x}) = 2.$$

Give the answer correct to 3 decimal places. [3]

$1 + e^{-3x} = e^2$ 1

$\cancel{e^2 - \frac{1}{e^{3x}} - 1 = 0}$

$\cancel{2}$

$\cancel{\ln 1 + \ln e}$ $\ln 1 - 3x = 2$ 2

$0 - \frac{3x}{-3} = \frac{2}{-3}$

$x = 0.667$

$x = -0.667$

Examiner comments

1 The candidate takes the correct first step, removing the logarithm.

2 The candidate uses an incorrect formula $\ln(A+B) = \ln A + \ln B$ so no further marks are awarded.

Total mark awarded = 1 out of 3

How the candidate could have improved their answer

The candidate could have paused to think about the basic rules for this topic. Use of an incorrect formula meant that no further marks could be awarded.

Common mistakes candidates made in this question

- Incorrect removal of brackets to give $\ln 1 + (-3x) = 2$.
- A correct first step followed by incorrect use of logarithms to give $\ln 1 + (-3x) = 2$.
- Calculator errors.
- Not giving the final answer to the required accuracy.

Question 2

Example Candidate Response – high	Examiner comments
2 (a) Expand $\sqrt[3]{1+6x}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]	
$\sqrt[3]{1+6x} = (1+6x)^{\frac{1}{3}}$ 1	1 The candidate carries out a correct conversion into index form.
$1 + (\frac{1}{3}\times 6x) + \frac{(\frac{1}{3})(\frac{1}{3}-1)(6x)^2}{2!} + \frac{(\frac{1}{3})(\frac{1}{3}-1)(\frac{1}{3}-2)(6x)^3}{3!}$ 2	2 The candidate gives a correct expansion in unsimplified form.
$1 + 2x - 4x^2 + \frac{40}{3}x^3$ 3	3 The accuracy marks are awarded when the candidate obtains a correct expansion in simplified form. Mark for (a) = 4 out of 4
(b) State the set of values of x for which the expansion is valid. [1]	
$\lvert 6x \rvert < 1$ 4	4 The candidate makes a correct statement about the values of $6x$, but the question asks for the values of x. Mark for (b) = 0 out of 1 **Total mark awarded = 4 out of 5**

How the candidate could have improved their answer

The candidate's approach was entirely correct but, at the end their response they needed to answer the question. They were asked about the values of x and their response was about values of $6x$.

Example Candidate Response – middle	Examiner comments
2 (a) Expand $\sqrt[3]{1+6x}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4] $(1+6x)^{\frac{1}{3}}$ (1) $= 1 + \frac{1}{3} \times 6x + \frac{(\frac{1}{3}(\frac{1}{3}-1)).6x^2}{2!}$ (2) $+ \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!} 6x^3$ (3) $= \cancel{4}\ 1 + 2x - \frac{2}{3}x^2 + \frac{10}{27}x^3$	(1) The candidate correctly converts the expression into index form. (2) The first two terms in the binomial expansion are correct. (3) The structure of the remaining terms is correct, but the candidate is using powers of x rather than powers of $6x$ so the third and fourth terms are incorrect. Mark for (a) = 2 out of 4
(b) State the set of values of x for which the expansion is valid. [1] $1 + 6x = 0$ $x = -\frac{1}{6}$ (4) $-\frac{1}{6} < x < \frac{1}{6}$	(4) The candidate identifies the critical value and states the correct interval. Mark for (b) = 1 out of 1 **Total mark awarded = 3 out of 5**

How the candidate could have improved their answer

The correct use of brackets would have helped the candidate to avoid their error. They wrote $6x^2$ and $6x^3$ when the correct answer came from $(6x)^2$ and $(6x)^3$.

Example Candidate Response – low	Examiner comments
2 **(a)** Expand $\sqrt[3]{1+6x}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4] $\sqrt[3]{1+6x} = (1+6x)^{\frac{3}{2}} = 1 + nx + \frac{n(n-1)}{2!}(x)^2 + \frac{n(n-1)(n-2)}{3!}(x)^3$ $(1+6x)^{3/2} = 1 + \frac{3}{2}(6x) + \frac{(\frac{3}{2})(\frac{3}{2}-1)}{2!}(6x)^2 + \frac{\frac{3}{2}(\frac{3}{2}-1)(\frac{3}{2}-2)}{3!}(6x)^3$ $= 1 + 9x + \frac{27}{2}x^2 + \frac{-27}{2}x^3$	1 The candidate attempts to express the term in index form, but makes an error. 2 The candidate expands their form correctly, but is not awarded any marks because they do not obtain the method mark at the start of the question. Mark for (a) = 0 out of 4
(b) State the set of values of x for which the expansion is valid. [1] $\lvert x \rvert < 1$ $\lvert 6x \rvert < 1$ $\lvert x \rvert < 1/6$	3 It is assumed that, at this point, the candidate is simply quoting a general validity statement. 4 The candidate gives a correct response. Mark for (b) = 1 out of 1 **Total mark awarded = 1 out of 5**

How the candidate could have improved their answer

The candidate misinterpreted the symbol for the cube root. This meant that no marks were available in part **(a)** because they did not answer the question set.

Common mistakes candidates made in this question

- In part **(a)**, not using the correct index.
- In part **(a)**, an expansion using powers of x rather than $6x$.
- In part **(a)**, errors in arithmetic when simplifying the coefficients.
- In part **(b)**, only considering one side of the inequality.
- In part **(b)**, stating $|x| < 1$ as a statement of a remembered result, without applying that result to the current expansion.
- In part **(b)**, repetition of all or part of the answer to part **(a)**.

Question 3

Example Candidate Response – high

3 The variables x and y satisfy the relation $2^y = 3^{1-2x}$.

(a) By taking logarithms, show that the graph of y against x is a straight line. State the exact value of the gradient of this line. [3]

$y \log 2 = (1-2x) \log 3$ 1

$y \log 2 = \log 3 - 2x \log 3$

$y \log 2 = -2x \log 3 + \log 3$ 2

gradient $= -2$

3

(b) Find the exact x-coordinate of the point of intersection of this line with the line $y = 3x$. Give your answer in the form $\frac{\ln a}{\ln b}$, where a and b are integers. [2]

$3x \ln 2 = -2x \ln 3 + \ln 3$

$3x \ln 2 + 2x \ln 3 = \ln 3$

$x(3 \ln 2 + 2 \ln 3) = \ln 3$

$x = \dfrac{\ln 3}{\ln 8 + \ln 9}$ 4

$x = \dfrac{\ln 3}{\ln 72}$ 5

Examiner comments

1 The candidate correctly uses logarithms as the first step to remove the indices.

2 The candidate removes the brackets and rearranges the equation, but they do not mention how they know that the equation represents a straight line.

3 The value identified by the candidate is not the gradient of the line.
Mark for (a) = 1 out of 3

4 The candidate makes x the subject of the equation.

5 The candidate states the correct answer in the form required by the question.
Mark for (b) = 2 out of 2

Total mark awarded = 3 out of 5

How the candidate could have improved their answer

- In part **(a)**, the candidate correctly used logarithms, removed brackets and rearranged the equation, but they did not address the demand to show that the graph was a straight line. For example, they could have compared their equation with the standard form $y = mx + c$. The value of m would have given the correct value for the gradient.
- The response to part **(b)** was correct and concise.

Example Candidate Response – middle

3 The variables x and y satisfy the relation $2^y = 3^{1-2x}$.

(a) By taking logarithms, show that the graph of y against x is a straight line. State the exact value of the gradient of this line. [3]

$$2^y = 3^{1-2x}$$

$$y \ln 2 = (1-2x)\ln 3 \quad 1$$

$$y = \frac{\ln 3\,(1-2x)}{\ln 2}$$

$$y = \frac{\ln 3}{\ln 2} - \frac{2x\ln 3}{\ln 2} \quad 2$$

$$\text{gradient} = \frac{2\ln 3}{\ln 2} \quad 3$$

(b) Find the exact x-coordinate of the point of intersection of this line with the line $y = 3x$. Give your answer in the form $\frac{\ln a}{\ln b}$, where a and b are integers. [2]

$$\frac{\ln 3}{\ln 2} - \frac{2\ln 3}{\ln 2}x = 3x \qquad x = \frac{\ln 3}{3\ln 2 + 2\ln 3}$$

$$3x + \frac{2\ln 3}{\ln 2}x = \frac{\ln 3}{\ln 2} \quad 4$$

$$\left(3 + \frac{2\ln 3}{\ln 2}\right)x = \frac{\ln 3}{\ln 2}$$

$$x = \frac{\ln 3}{\cancel{\ln 2}} \times \frac{\cancel{\ln 2}}{3\ln 2 + 2\ln 3}$$

Examiner comments

1 The candidate uses a correct method to remove the indices.

2 The candidate rearranges the equation. At this stage, they could compare their equation with $y = mx + c$ to confirm the linear form, but there is no mention of this.

3 The candidate identifies the correct term for the gradient, but the answer should be negative not positive.
Mark for (a) = 1 out of 3

4 The candidate rearranges the equation correctly to obtain the value of x. The question asked for the value to be given in a particular form, and their denominator does not fulfil that. They need to show further steps in the working to be awarded full marks.
Mark for (b) = 1 out of 2

Total mark awarded = 2 out of 5

How the candidate could have improved their answer

- The response in part **(a)** was initially correct, but finished with a sign error in the gradient. If the candidate had shown the match between their equation and $y = mx + c$, the error might have been avoided. They would also have demonstrated that the graph was a straight line.
- The work in part **(b)** was correct but, to be awarded full marks, the answer needed to be given in the form requested.

Example Candidate Response – low

3 The variables x and y satisfy the relation $2^y = 3^{1-2x}$.

(a) By taking logarithms, show that the graph of y against x is a straight line. State the exact value of the gradient of this line. [3]

$2^y = 3^{1-2x}$

$2^y = \cancel{8}\ \frac{3^1}{3^{2x}}$ 1

$\ln(2^y) = \ln\left(\frac{3^1}{3^{2x}}\right)$

$y\ln 2 = \ln\left(\frac{3}{3^{2x}}\right)$

$y\ln 2 = \ln 3 - \ln(3^{2x})$

$y\ln 2 = \ln 3 - 2x\ln 3$

$y = \frac{\ln 3}{\ln 2} - \frac{2x\ln 3}{\ln 2}$ 2

$y = x\frac{2\ln 3}{\ln 2} - \frac{\ln 3}{\ln 2}$

The value of the gradient is $\frac{2\ln 3}{\ln 2}$. 3

(b) Find the exact x-coordinate of the point of intersection of this line with the line $y = 3x$. Give your answer in the form $\frac{\ln a}{\ln b}$, where a and b are integers. [2]

$y = \frac{2\ln 3}{\ln 2}x - \frac{\ln 3}{\ln 2}$, $y = 3x$

4 $3x = \frac{2\ln 3}{\ln 2}x - \frac{\ln 3}{\ln 2}$

$3 = \frac{2\ln 3}{\ln 2} - \frac{\ln 3}{x\ln 2}$

$\frac{\ln 3}{x\ln 2} = \frac{2\ln 3}{\ln 2}$

$\frac{\ln 3}{x} = \frac{2\ln 3\,\cancel{\ln 2}}{\cancel{\ln 2}}$

$\frac{\ln 3}{x} = 2\ln 3$

$\ln 3 = x\,2\ln 3$

$\frac{\ln 3}{2\ln 3} = x$

$\frac{3+\ln 3}{x\ln 2} = \frac{2\ln 3}{2}$ 5

$3+\ln 3 = \ln 3\,(x\ln 2)$

$\frac{3+\ln 3}{\ln 3} = \ln 2 \cdot x$

$\frac{3+\ln 3}{\ln 2\ln 3} = x$

Examiner comments

1 The candidate's working is correct. It would be acceptable to go directly from the form in the question to their fifth line of working.

2 This is a correct equation. The candidate was asked to show that the graph will be a straight line so they need to comment that this equation is of the form $y = mx + c$.

3 A sign error in rearranging the equation leads to an incorrect value for the gradient.
Mark for (a) = 1 out of 3

4 The candidate starts with a correct statement based on their version of the equation.

5 The candidate makes errors in the algebra at this stage of the working, so the method mark is not awarded.
Mark for (b) = 0 out of 2

Total mark awarded = 1 out of 5

How the candidate could have improved their answer

- The candidate gave correct working, but their response did not address the request to show that the graph was a straight line. They needed to say how they knew that it was a straight line, for example, they could have compared their equation with $y = mx + c$. The sign error at the end of part **(a)** might have been detected through careful checking at the end of the exam.
- In part **(b)**, the candidate made algebraic errors. For a small slip, the method mark could have been awarded. However, the errors in dealing with the fractions meant no further marks were awarded.

Common mistakes candidates made in this question

- In part **(a)**, omission of the brackets when first taking logarithms, obtaining $y \ln 2 = 1 - 2x \ln 3$.
- In part **(a)**, no statement to explain how candidates knew that the graph would be a straight line.
- In part **(a)**, an equation with subject $y \ln 2$ followed by a statement that the gradient was $\pm 2 \ln 3$ or $\pm 2x \ln 3$.
- In part **(a)**, confusion between the gradient and the intercept on the y-axis.
- In part **(a)**, giving a decimal approximation to the gradient when the question asked for the exact value.
- In part **(b)**, algebraic errors in the course of making x the subject of the equation.
- In part **(b)**, incorrect manipulation of the logarithms when rearranging the equation.
- In part **(b)**, the use of decimal approximations when the question asked for an exact answer.
- In part **(b)**, working in alternative bases for logarithms and not converting the final answer to the required form.
- In part **(b)**, not taking the final step to express $2 \ln 3 + 3 \ln 2$ as a single logarithm.
- In part **(b)**, the use of incorrect laws in simplifying $2 \ln 3 + 3 \ln 2$.

Question 4

Example Candidate Response – high

4 (a) Show that the equation $\tan(\theta + 60°) = 2\cot\theta$ can be written in the form

$$\tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0. \qquad [3]$$

$\tan(\theta + 60) = 2\dfrac{1}{\tan\theta}$

$\dfrac{\tan\theta + \tan 60}{1 - \tan\theta\tan 60}$

$\dfrac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta} = \dfrac{2}{\tan\theta}$ (1)

$\tan^2\theta + \sqrt{3}\tan\theta = 2 - 2\sqrt{3}\tan\theta$

(2) $\tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0$

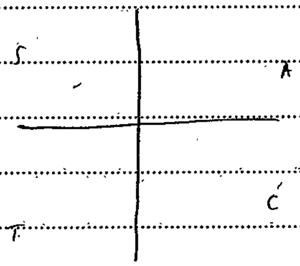
(b) Hence solve the equation $\tan(\theta + 60°) = 2\cot\theta$, for $0° < \theta < 180°$. [3]

$\tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0$

(3) $\tan\theta = 0.36 \quad \theta = 19.8$

$\tan\theta = -5.56 \quad \theta = 79.8$

$(+) \quad (-)$



$\theta = 19.8 \quad \theta = 79.8$

$\theta = \quad \theta = 100.2$

(4)

Examiner comments

(1) The candidate states a correct equation in terms of $\tan\theta$.

(2) The candidate rearranges their equation to obtain the given form.
Mark for (a) = 3 out of 3

(3) The candidate states correct values for $\tan\theta$.

(4) The candidate states two correct values for the angle. They also include the incorrect value 79.8. This value is correctly obtained in their working, but should not be part of the final answer. Consequently, the final mark is not awarded.
Mark for (b) = 2 out of 3

Total mark awarded = 5 out of 6

How the candidate could have improved their answer

Most of the work here was correct. At the end, the candidate could have distinguished between values used as part of the working and the final answer. The candidate included an incorrect value in their final answer so could not be awarded the final mark.

Example Candidate Response – middle	**Examiner comments**
4 (a) Show that the equation $\tan(\theta + 60°) = 2\cot\theta$ can be written in the form $$\tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0.$$ [3]	
$\tan(\theta + 60°) = 2\cot\theta$	
$\Rightarrow \dfrac{\tan\theta + \cancel{\tan 60°}\sqrt{3}}{1 - \tan\theta\,\cancel{\tan 60°}\sqrt{3}} = \dfrac{2}{\tan\theta}$	
1 $\Rightarrow \dfrac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\cdot\tan\theta} = \dfrac{2}{\tan\theta}$	1 The candidate obtains a correct equation in $\tan\theta$.
$\Rightarrow \tan\theta(\tan\theta + \sqrt{3}) = 2(1 - \sqrt{3}\tan\theta)$ $\Rightarrow \tan^2\theta + \sqrt{3}\tan\theta = 2 - 2\sqrt{3}\tan\theta$ 2 $\Rightarrow \tan^2\theta + 2\sqrt{3}\tan\theta + \sqrt{3}\tan\theta - 2 = 0$ $\Rightarrow \tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0$ (shown)	2 The candidate rearranges the equation correctly to obtain the given form. Mark for (a) = 3 out of 3
(b) Hence solve the equation $\tan(\theta + 60°) = 2\cot\theta$, for $0° < \theta < 180°$. [3]	
$\tan(\theta + 60°) = 2\cot\theta$ for $0° < \theta < 180°$	
$\Rightarrow \tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0$ $\Rightarrow \tan^2\theta + 3(3)^{1/2}\tan\theta - 2 = 0$ let $u = \tan\theta$; 3 $\therefore u^2 + 3\sqrt{3}u - 2 = 0$ $u^2 + 3\sqrt{3}u = 2$ $u(u + 3\sqrt{3}) = 2$	3 The candidate recognises the equation is a quadratic equation and substitutes to obtain simpler notation.
4 $u = 2$ $\quad u + 3\sqrt{3} = 2$ $u = 2 - 3\sqrt{3}$ But $u = \tan\theta$; $\tan\theta = 2$ $\quad \tan\theta = 2 - 3\sqrt{3}$ $\theta = \tan^{-1}(2)$ $\quad \theta = \tan^{-1}(2 - 3\sqrt{3})$ $\theta = 63.4°$ p.v. $\quad \theta = \cancel{-73.6°}$	4 The correct method is to consider the factors of zero. Nothing can be deduced by considering the factors of 2, so the candidate is not using a valid method and no further marks are awarded. Mark for (b) = 0 out of 3
$\theta = 63.4°$	**Total mark awarded = 3 out of 6**

How the candidate could have improved their answer

- The response to part **(a)** was correct and concise yet showed all the required working.
- Part **(b)** started correctly and the candidate reached a correct quadratic equation. At this point, candidates needed to solve the quadratic equation by considering factors of zero. Nothing could be determined by considering the factors of 2.

Example Candidate Response – low	Examiner comments

4 **(a)** Show that the equation $\tan(\theta + 60°) = 2\cot\theta$ can be written in the form

$$\tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0. \qquad [3]$$

to show that $\tan(\theta + 60°) = 2\cot\theta \equiv$
$\tan^2\theta + 3\sqrt{3}\tan - 2 = 0$ 1

LHS $\tan(\theta + 60°) = \dfrac{1}{2\tan\theta} \equiv \tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0$

$\tan(\theta + 60°) = \dfrac{\tan\theta + \tan 60°}{1 - \tan\theta\tan 60°}$ 2

$\therefore \dfrac{\tan\theta + \tan 60°}{1 - \tan\theta\tan 60°} - \dfrac{1}{2\tan\theta} = 0 \equiv \tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0$

because $\tan 60° = \sqrt{3}$

$\dfrac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta} - \dfrac{1}{2\tan\theta} = 0$

3

$(2\tan\theta)(\tan\theta + \sqrt{3}) - 1(1 - \sqrt{3}\tan\theta) = 0$

$2\tan^2\theta + 2\tan\theta\sqrt{3}) - 1 - \sqrt{3}\tan\theta = 0$

$2(\tan^2\theta\ \tan\theta\sqrt{3}) - 1 - \sqrt{3}\tan\theta = 0$

4

$\therefore \tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0 \equiv \tan^2\theta + 3\sqrt{3}$ ...

shown

Examiner comments

1 The candidate attempts to write the right-hand side in terms of $\tan\theta$ but the 2 should have remained in the numerator.

2 The candidate uses the expansion of $\tan(A+B)$ correctly.

3 The error prevents the candidate obtaining a correct equation free of fractions.

4 The candidate attempts to rearrange their equation to obtain the given form but the earlier error makes this impossible.
Mark for (a) = 1 out of 3

Example Candidate Response – low, continued

(b) Hence solve the equation $\tan(\theta + 60°) = 2\cot\theta$, for $0° < \theta < 180°$. [3]

Complete 5

$\tan(\theta + 60°) = 2\cot\theta = \tan^2\theta + 3\sqrt{3}\tan\theta - 2 =$

$\therefore \tan^2\theta + 3\sqrt{}\tan\theta - 2 = 0$

let $\sqrt{\tan\theta} = y$ $\quad -\frac{c}{s}$

$\therefore y^2 + 3y - 2 = 0$ 6

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-2)}}{2(1)}$

$x = \frac{-3 \pm \sqrt{17}}{2}$ 7

$x = 0,5612$ or $-3,562$

So $y = \sqrt{\tan\theta}$ 8

$(\sqrt{\tan\theta})^2 = (0,5612)^2$ or $-3,562$

$\theta = \tan^{-1}(0,3149)$ or $12,689$

$\theta = 17,47$ or $85,49$

S	A
T	C

$\theta = 17,47°, 162,5$ or $85,5$

Examiner comments

5 The candidate states the equation correctly here.

6 The candidate recognises the equation as a quadratic in $\tan\theta$ but they make an error in substituting y for $\tan\theta$.

7 The candidate uses the quadratic formula correctly for their equation.

8 The candidate introduces an incorrect square root sign. Because they square each value before using the inverse trig function, they do not have a correct method.
Mark for (b) = 0 out of 3

Total mark awarded = 1 out of 6

How the candidate could have improved their answer

The candidate might have avoided some errors by checking the algebra carefully. In the first part, they used $\cot\theta = \frac{1}{\tan\theta}$ but they also replaced 2 with $\frac{1}{2}$. In the second part, they recognised the quadratic equation, but in attempting to simplify it, they have changed $3\sqrt{3} \to 3\sqrt{\ } \to 3$ and consequently were awarded no marks.

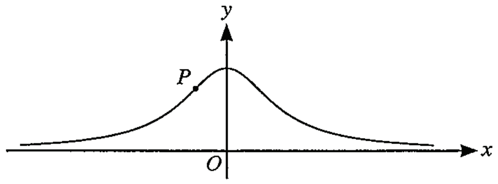
Common mistakes candidates made in this question

- In part **(a)**, expanding $\tan(\theta + 60°)$ as $\tan\theta + \tan 60°$.
- In part **(a)**, rewriting $2\cot\theta$ as $\frac{1}{2\tan\theta}$.
- In part **(b)**, premature approximation of decimal values.
- In part **(b)**, giving incorrect over-specified answers.
- In part **(b)**, the inclusion of additional incorrect solutions.
- In part **(b)**, concluding that $\tan\theta = -5.556$ gave no solution within the range.
- In part **(b)**, incorrect attempts to solve a quadratic equation, e.g. writing the given equation as $\tan\theta(\tan\theta + 3\sqrt{3}) = 2$ and equating each factor to 2.

Question 5

Example Candidate Response – high	Examiner comments

5



The diagram shows the curve with parametric equations

$$x = \tan\theta, \quad y = \cos^2\theta, \qquad (\cos\theta)^2.$$

for $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. $\qquad 2(\cos\theta)(-\sin\theta)$

(a) Show that the gradient of the curve at the point with parameter θ is $-2\sin\theta\cos^3\theta$. [3]

$\frac{dx}{d\theta} = \sec^2\theta$; $\frac{dy}{d\theta} = -2\sin\theta\cos\theta$. 1

$\frac{dy}{d\theta} = \frac{dy}{dx} \times \frac{dx}{d\theta}$

$-2\sin\theta\cos\theta = \frac{dy}{dx}$ 2 $\times \sec^2\theta$.

$-2\sin\theta\cos\theta \div \frac{1}{\cos^2\theta} = \frac{dy}{dx}$

$-2\sin\theta\cos^3\theta = \frac{dy}{dx}$ 3

Examiner comments

1 The candidate states correct values for both derivatives.

2 The candidate uses the correct method to find the gradient of the curve.

3 The candidate obtains the given answer correctly. Mark for (a) = 3 out of 3

Example Candidate Response – high, continued

The gradient of the curve has its maximum value at the point P.

(b) Find the exact value of the x-coordinate of P. [4]

At Point P, $\frac{d^2y}{dx^2} = 0$.

$\cancel{+2\sin\phi\cos^3\phi / + / \phi}$.

$\frac{d^2y}{dx^2} = uv' + vu'$.

$u = -2\sin\theta$; $v = \cos^3\theta$.

$u' = -2\cos\theta$. $v' = 3(\cos\theta)^2(-\sin\theta)$

$v' = -3\sin\theta\cos^2\theta$.

$\frac{d^2y}{dx^2} = (-2\sin\theta)(-3\sin\theta\cos^2\theta) + (\cos^3\theta)(-2\cos\theta)$

4

$0 = 6\sin^2\theta\cos^2\theta + (-2\cos^4\theta)$.

$0 = 2\cos^2\theta(3\sin^2\theta - \cos^2\theta)$.

$2\cos^2\theta = 0$ or $3\sin^2\theta - \cos^2\theta = 0$

Rejected $3\sin^2\theta = \cos^2\theta$

5 $3\tan^2\theta = 1$.

$\tan\theta = \sqrt{\frac{1}{3}}$ 6

$\theta = \pi/6$.

x-coordinate of P:-

$x = \tan(\pi/6)$

$\boxed{x = \frac{\sqrt{3}}{3}}$ 7

Examiner comments

4 The candidate uses the product rule correctly to differentiate the gradient.

5 This is a correct decision. This factor gives no solutions in the required range.

6 The candidate is solving the correct equation but overlooks the negative root.

7 The candidate reaches an incorrect conclusion that does not match the graph.
Mark for (b) = 3 out of 4

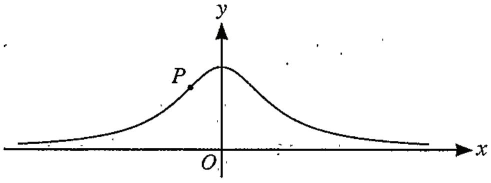
Total mark awarded = 6 out of 7

How the candidate could have improved their answer

The candidate could have checked the diagram for information relevant to the final answer. The graph clearly showed that the x-coordinate of P was negative but, at the end of their solution, the candidate has only considered the positive root of their equation and they could not be awarded the final mark.

Example Candidate Response – middle

5



The diagram shows the curve with parametric equations

$$x = \tan\theta, \quad y = \cos^2\theta,$$

for $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

(a) Show that the gradient of the curve at the point with parameter θ is $-2\sin\theta\cos^3\theta$. [3]

$x = \tan\theta$ $\qquad \frac{dy}{d\theta} = 2\cos\theta\sin\theta$ 2

1 $\frac{dx}{d\theta} = \sec^2\theta$

$= \frac{1}{\cos^2\theta}$

$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = -2\cos\theta\sin\theta$ 3

$= -2\cos^3\theta\sin\theta$

4

The gradient of the curve has its maximum value at the point P.

(b) Find the exact value of the x-coordinate of P. [4]

$v = \cos^2\theta$ 5 $\qquad u = -2\sin\theta$

$\frac{dv}{d\theta} = -3\cos^2\theta\sin\theta$ $\qquad \frac{du}{d\theta} = -2\cos\theta$

$= \cos^3\theta - 2\cos\theta + 3\cos^2\theta\sin\theta - 2\sin\theta$ 6

$= -2\cos^4\theta + 6\sin^2\theta\cos^2\theta = 0$

$+2\cos^2\theta(-\cos^2\theta + 3\sin^2\theta) = 0$ 8

7 $2\cos^2\theta = 0, \quad \theta = \frac{1}{2}\pi$

Examiner comments

1 The derivative of x is correct.

2 The derivative of y should be negative. It is difficult to be sure whether or not this expression contains a '–' sign, but only one of the derivatives needs to be correct to be awarded the first mark.

3 The candidate states the correct method but there is no evidence that they use it. The method mark is not available.

4 The candidate states the given answer, but there is no working to support it. Mark for (a) = 1 out of 3

5 There is a slip here, but the correct derivative implies that the candidate uses the correct function.

6 There is a possible sign error in the second term of the derivative, but the next line confirms that the candidate uses the product rule correctly.

7 This value falls outside the interval given in the question.

8 The candidate does not consider possible solutions from this bracket. Mark for (b) = 2 out of 4

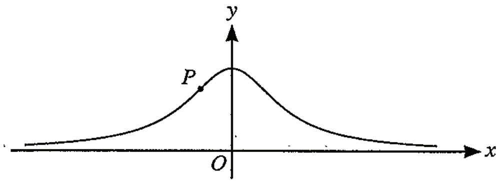
Total mark awarded = 3 out of 7

How the candidate could have improved their answer

- Since the question began with 'Show that ...' the candidate needed to show fully correct working. In part **(a)**, it looked as if they made a sign error in the top line, and the lack of a clear substitution into the correct formula made it impossible to be sure. This candidate has concluded with the required answer but they have not reached it from their working.
- The candidate could have checked their answers carefully to avoid the two slips in the first line of part **(b)**. Although they wrote $\cos^2\theta$ in place of $\cos^3\theta$, their following work suggested that they intended the correct term. They could have detected the sign error in the derivative by reference to the list of standard results given in the formula booklet.

Example Candidate Response – low

5



The diagram shows the curve with parametric equations

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for $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

(a) Show that the gradient of the curve at the point with parameter θ is $-2\sin\theta\cos^3\theta$. [3]

$\frac{dx}{dt} = \sec^2\theta$ 1 $\quad \cancel{\frac{dy}{dt}} \quad y = 2\cos\theta$ 2

$\frac{dy}{dt} = \cancel{-\sin x} \; -2\sin\theta$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

$= -2\sin\theta \times \frac{1}{\sec^2\theta} \qquad \sec^2\theta = 1 + \tan^2\theta$

3

$= -2\sin\theta \times \frac{1}{(1 + \tan^2\theta)}$

$= -2\sin\theta \times \frac{1}{\left(1 + \cancel{\cos^2\theta}\frac{\sin^2\theta}{\cos^2\theta}\right)}$

$= -2\sin\theta \times 1 + \frac{\cos^2\theta}{\sin^2\theta}$

$= -2\cancel{\sin\theta} \times \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}$

4

$= \frac{-2\sin^2\theta \; \cancel{\cos\theta} \; + 2\cos^2\theta}{\cancel{\sin\theta}} \Rightarrow -2\sin\theta$

$= \frac{-2(1)}{\sin\theta} = \frac{-2}{\sin\theta}$

The gradient of the curve has its maximum value at the point P.

(b) Find the exact value of the x-coordinate of P. [4]

$\frac{dy}{dx} = -2\sin\theta\cos^3\theta$

5

Examiner comments

1 The candidate provides a correct derivative.

2 This is not the expression for y given in the question, and the change in form suggests that the candidate does not misread the information in the question. Only one of the two derivatives needs to be correct to obtain the first mark.

3 The candidate states the correct method for finding the gradient, and they use their two derivatives correctly.

4 The candidate's earlier error means that they cannot reach the given result.
Mark for (a) = 2 out of 3

5 The candidate quotes the correct form of the gradient, but there is no indication of a strategy to answer the question.
Mark for (b) = 0 out of 4

Total mark awarded = 2 out of 7

How the candidate could have improved their answer

- In part **(a)**, the candidate overlooked the simple substitution $\frac{1}{\sec^2\theta}=\cos^2\theta$ and took a more complicated route in which they did not recognise $\cos^2\theta+\sin^2\theta=1$. They could have checked first whether the expression could be simplified before making it more complicated.
- In part **(b)**, the hint was in the rubric to the question: the task was to locate the maximum value of the function, so the first step was to differentiate.

Common mistakes candidates made in this question

- In part **(a)**, incorrect differentiation of $\cos^2\theta$, e.g. stating an answer $\pm\sin^2\theta$.
- In part **(a)**, inconsistent signs in the working.
- In part **(b)**, using an approximate method of solution (e.g., considering tabulated values of the function) when the question asks for an exact answer.
- In part **(b)**, looking for the point(s) where the function itself was zero, rather than where the gradient of the function was zero.
- In part **(b)**, errors in differentiating $\cos^3\theta$.
- In part **(b)**, giving a positive final answer although the diagram shows P as a point with a negative x-coordinate.

Question 6

Example Candidate Response – high

6 The complex number u is defined by

$$u = \frac{7+\mathrm{i}}{1-\mathrm{i}}.$$

(a) Express u in the form $x + \mathrm{i}y$, where x and y are real. [3]

$$u = \frac{7+i}{1-i}$$

$$\frac{7+i}{1-i} \times \frac{1+i}{1+i}$$ 1

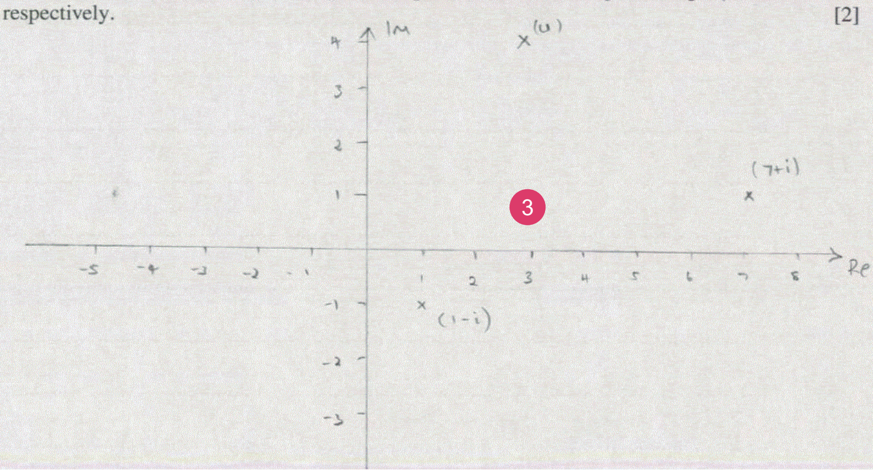
$$\frac{(7+i)(1+i)}{(1-i)(1+i)}$$

$$\frac{7+7i+i+i^2}{1+i-i-i^2}$$ 2

$$\frac{7+8i-1}{1+1}$$

$$\frac{6+8i}{2} \rightarrow 3+4i$$

(b) Show on a sketch of an Argand diagram the points A, B and C representing u, $7+\mathrm{i}$ and $1-\mathrm{i}$ respectively. [2]



Examiner comments

1 The candidate uses the correct method.

2 The candidate's working is correct and simplifies to the correct answer.
Mark for (a) = 3 out of 3

3 The candidate plots all three points correctly.
Mark for (b) = 2 out of 2

Example Candidate Response – high, continued	Examiner comments
(c) By considering the arguments of $7+\mathrm{i}$ and $1-\mathrm{i}$, show that $$\tan^{-1}\left(\tfrac{4}{3}\right)=\tan^{-1}\left(\tfrac{1}{7}\right)+\tfrac{1}{4}\pi.$$ [3]	
$7+i$ and $1-i$ $\quad \tan\theta = \frac{y}{x}$ $\tan\theta = \left(\frac{1}{7}\right) \quad \tan\theta = \left(\frac{-1}{1}\right)$ $\theta = \tan^{-1}\left(\frac{1}{7}\right) \quad \theta = \tan^{-1}\left(\frac{-1}{1}\right)$ $= -\frac{1}{4}\pi$ 4	4 This is a correct statement about the argument of $1-\mathrm{i}$.
$\tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(\frac{1}{7}\right) + \frac{1}{4}\pi$ 5	5 The candidate writes down the given answer, but they give no explanation of how this is obtained. Mark for (c) = 1 out of 3 **Total mark awarded = 6 out of 8**

How the candidate could have improved their answer

- The candidate gave fully correct answers to parts **(a)** and **(b)**.
- Part **(c)** required full working because the request was to 'show that ...'. The candidate has noted the instruction for the first step but, to make further progress, they needed to recognise how part **(c)** was related to part **(a)**, or else work through from first principles.

Example Candidate Response – middle

6 The complex number u is defined by

$$u = \frac{7+i}{1-i}.$$

(a) Express u in the form $x + iy$, where x and y are real. [3]

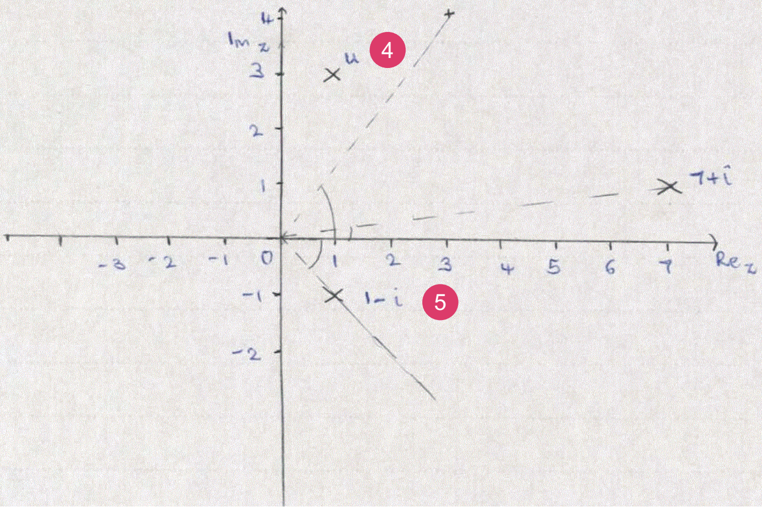
$u = \dfrac{7+i}{1-i} \times \dfrac{(1+i)}{(1+i)}$ 1

$= \dfrac{7+i+i-1}{1+1}$ 2

$= \dfrac{6+2i}{2}$

$u = 3+i$ 3

(b) Show on a sketch of an Argand diagram the points A, B and C representing u, $7 + i$ and $1 - i$ respectively. [2]



(c) By considering the arguments of $7 + i$ and $1 - i$, show that

$$\tan^{-1}\left(\tfrac{4}{3}\right) = \tan^{-1}\left(\tfrac{1}{7}\right) + \tfrac{1}{4}\pi.$$ [3]

$\text{Arg}(7+i) = \tan^{-1}\left(\frac{1}{7}\right)$

~~Arg $\left(-\frac{1}{1}\right)$~~

$\text{Arg}(1-i) = \tan^{-1}\left(-\frac{1}{1}\right)$

$= -\frac{1}{4}\pi$ 6

7 $\tan^{-1}\frac{4}{3} = \tan^{-1}\frac{1}{7} + \frac{1}{4}\pi$

Examiner comments

1 The candidate selects the correct method.

2 This is sufficient evidence that the candidate is using a correct method for their multiplication, but there is an error in the numerator.

3 As a result of the candidate's error, the final answer is not correct.
Mark for (a) = 2 out of 3

4 The candidate has set up the Argand diagram correctly but this point represents $1+3i$, not their u.

5 The candidate plots the other two points correctly.
Mark for (b) = 1 out of 2

6 This is a correct argument.

7 The candidate states the given result but does not explain how they obtained it.
Mark for (c) = 1 out of 3

Total mark awarded = 4 out of 8

How the candidate could have improved their answer

- Through careful checking, the candidate could have detected both errors in the first two parts. In part **(a)**, the candidate has $7 \times 1 = 1$ and in part **(b)**, they have plotted $1 + 3\text{i}$ in place of $3 + \text{i}$.
- In part **(c)**, there was no working to support the given answer, so perhaps they have not recognised the link between part **(a)** and part **(c)**.

Example Candidate Response – low

6 The complex number u is defined by

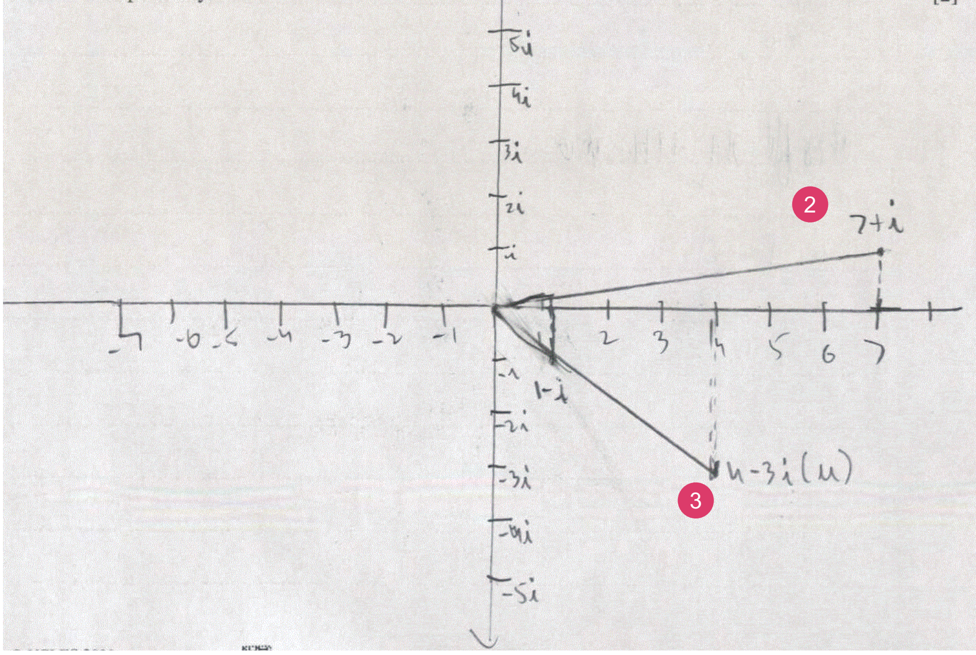
$$u = \frac{7+\mathrm{i}}{1-\mathrm{i}}.$$

(a) Express u in the form $x + \mathrm{i}y$, where x and y are real. [3]

$$u = \frac{7+i}{1-i} \times \frac{1-i}{1-i} = \frac{7-7i+i+1}{2}$$

$$\Rightarrow u = \frac{8-6i}{2} = 4-3i$$

(b) Show on a sketch of an Argand diagram the points A, B and C representing u, $7+\mathrm{i}$ and $1-\mathrm{i}$ respectively. [2]



Examiner comments

1 This incorrect form, and the denominator 2 that follows, suggest that the candidate has some knowledge of how to divide complex numbers but they apply the method incorrectly. The method mark is not awarded.
Mark for (a) = 0 out of 3

2 The candidate plots the other two points correctly.

3 The candidate correctly plots the point representing their answer to part **(a)**, so they are awarded the follow-through mark.
Mark for (b) = 2 out of 2

Example Candidate Response – low, continued

(c) By considering the arguments of $7+\mathrm{i}$ and $1-\mathrm{i}$, show that

$$\tan^{-1}\left(\tfrac{4}{3}\right) = \tan^{-1}\left(\tfrac{1}{7}\right) + \tfrac{1}{4}\pi. \qquad [3]$$

$z_1 = 7 + i = r\cos\theta + i r\sin\theta$

$\Rightarrow 7 = \cos\theta$

$1 = \sin\theta$

$\Rightarrow \tan\theta = \frac{1}{7}$

$\Rightarrow \theta = \arg(z_1) = \tan^{-1}\left(\frac{1}{7}\right)$

$z_2 = 1 - i = r\cos\phi + i r\sin\phi$

$\Rightarrow \tan\phi = -1$

$\Rightarrow \phi = \arg(z_2) = -\frac{\pi}{4}$ 4

~~arg(u)~~

$\alpha = \arg(u) = \tan^{-1}\left(\frac{-4}{3}\right)$

5

$\Rightarrow \tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(\frac{1}{7}\right) + \frac{1}{4}\pi$ 6

Examiner comments

4 The candidate correctly states the value of $\arg(1-\mathrm{i})$.

5 The question gives the hint that the route to the given result is by considering the arguments of the complex numbers, but the candidate does not form an equation linking the arguments.

6 The candidate copies the given answer which does not follow a correct line of reasoning.
Mark for (c) = 1 out of 3

Total mark awarded = 3 out of 8

How the candidate could have improved their answer

- In part **(a)**, the candidate made an error which could have been detected if they had taken care to expand $(1-\mathrm{i})^2$ properly.
- In part **(b)**, they demonstrated correct plotting of points on an Argand diagram.
- The error in part **(a)** meant that their diagram did not help them with part **(c)**. As an alternative, they could have used the expansion of $\tan(A-B)$ with angles $\tan^{-1}\left(\frac{4}{3}\right)$ and $\tan^{-1}\left(\frac{1}{7}\right)$.

Common mistakes candidates made in this question

- In part **(a)**, no indication of method.
- In part **(a)**, the use of $1-\mathrm{i}$ in place of $1+\mathrm{i}$.
- In part **(a)**, slips in the arithmetic and incorrect simplification of i^2.
- In part **(a)**, some candidates who adopted the alternative method of multiplying both sides of the equation by $1-\mathrm{i}$ did not go on to consider the real and imaginary parts of their equation.
- In part **(b)**, diagrams with uneven scales.
- In part **(b)**, incorrect plotting of points.
- In part **(b)**, confusing the real and imaginary axes.
- In part **(c)**, stating that $\arg(1-\mathrm{i}) = \frac{\pi}{4}$.
- In part **(c)**, no statement linking the arguments of the complex numbers.
- In part **(c)**, working implying that in the division of two complex numbers the arguments are added.
- In part **(c)**, the use of decimal approximations to establish an exact answer.

Question 7

Example Candidate Response – high

7 The variables x and t satisfy the differential equation

$$e^{3t}\frac{dx}{dt} = \cos^2 2x,$$

for $t \geqslant 0$. It is given that $x = 0$ when $t = 0$.

(a) Solve the differential equation and obtain an expression for x in terms of t. [7]

$e^{3t}\frac{dx}{dt} = \cos^2 2x$ 1

$\Rightarrow \int \frac{1}{\cos^2 2x}\,dx = \int \frac{1}{e^{3t}}\,dt$ $\left[\cos^2\theta = \frac{1}{\sec^2\theta}\right]$

$\Rightarrow \int \sec^2 2x\,dx = \int e^{-3t}\,dt$

$\Rightarrow \frac{1}{2}\tan 2x = -\frac{1}{3}e^{-3t} + c'$

$\Rightarrow$ when $x=0$, $t=0$, 2

$\frac{1}{2}\tan(0) = -\frac{1}{3}e^{0} + c$

$\Rightarrow 0 = -\frac{1}{3} + c$

$\Rightarrow c = \frac{1}{3}$

$\therefore \boxed{c = \frac{1}{3}}$ 3

$\frac{1}{2}\tan 2x = -\frac{1}{3}e^{-3t} + \frac{1}{3}$

$\Rightarrow \tan 2x = -\frac{2}{3e^{3t}} + \frac{1}{3}$ 4

$\Rightarrow \tan 2x = \frac{1}{3} - \frac{2}{3e^{3t}}$

$\Rightarrow x = \frac{1}{2}\tan^{-1}\left(\frac{1}{3} - \frac{2}{3e^{3t}}\right)$ Ans.

~~$\Rightarrow x = \frac{1}{2}\tan^{-1}($~~ 5

(b) State what happens to the value of x when t tends to infinity. [1]

$t \to \infty$, $\frac{2}{3e^{3t}} \to 0$

$x \to \frac{1}{2}\tan^{-1}\left(\frac{1}{3}\right)$ 6

Examiner comments

1 The candidate separates the variables correctly.

2 The candidate completes both integrals correctly.

3 The candidate obtains the correct value for the constant of integration.

4 There is an error in multiplying through by 2.

5 The final answer is not correct due to the earlier error.
Mark for (a) = 6 out of 7

6 The candidate reaches the correct conclusion for their equation.
Mark for (b) = 1 out of 1

Total mark awarded = 7 out of 8

How the candidate could have improved their answer

The methods used by the candidate in this response were all clear and correct. By careful checking at the end, they might have found the error in multiplying by 2. However, despite this error, because their answer had the correct structure, they obtained the 'follow-through' mark in part **(b)**.

Example Candidate Response – middle	Examiner comments
7 The variables x and t satisfy the differential equation $$e^{3t}\frac{dx}{dt} = \cos^2 2x,$$ for $t \geqslant 0$. It is given that $x = 0$ when $t = 0$. **(a)** Solve the differential equation and obtain an expression for x in terms of t. [7] $x=0,\ t=0 \quad t \geq 0$ $e^{3t}\frac{dx}{dt} = \cos^2 2x$ $\int \frac{1}{\cos^2 2x}\,(dx) = \int \frac{1}{e^{3t}}\,(dt)$ **1** $\int \frac{1}{(\cos 2x)^2}\,(dx) = \int e^{-3t}\,(dt)$ $\int \frac{1}{(\cos 2x)^2}\,(dx) = \frac{e^{-3t}}{-3} + c$ **2** $\int (\sec 2x)^2\,(dx) = \frac{e^{-3t}}{-3} + c$ $\int \sec^2 2x\,(dx) = \frac{e^{-3t}}{-3} + c$ **3** $\tan 2x = \frac{e^{-3t}}{-3} + c$ $t=0,\ x=0$ $\tan(2\times 0) = -\frac{e^{-3(0)}}{3} + c$ $0 = -\frac{1}{3} + c$ $c = \frac{1}{3}$ **4** $\tan 2x = -\frac{e^{-3t}}{3} + \frac{1}{3}$	**1** The candidate separates the variables correctly. **2** The integration of the right-hand side is correct. **3** The candidate recognises the form of the left-hand side and uses a correct method of integration. They use the correct function, but not the correct coefficient. **4** The candidate uses the limits correctly to find the constant of integration.

Example Candidate Response – middle, continued	Examiner comments
$\tan 2x = \frac{1}{3}(-e^{-3t}+1)$ $2x = \tan^{-1}\left(\frac{1}{3}(-e^{-3t}+1)\right)$ $x = \frac{\tan^{-1}\left(\frac{1}{3}(-e^{-3t}+1)\right)}{2}$ 5	5 The process for finding an expression for x in terms of t is correct, but no accuracy marks are awarded due to the earlier accuracy error. Mark for (a) = 4 out of 7
(b) State what happens to the value of x when t tends to infinity. [1] x tends to stay $\tan^{-1}\left(\frac{1}{3}\right) \Rightarrow \frac{\ }{2}$ 6	6 The candidate uses their answer to part **(a)** and obtains the corresponding limit. There is no working or explanation: the correct conclusion is sufficient. Mark for (b) = 1 out of 1 **Total mark awarded = 5 out of 8**

How the candidate could have improved their answer

The candidate took the crucial step in recognising $\frac{1}{\cos^2 2x}$ as $\sec^2 2x$. When dealing with a ‘function of a function’ they needed to take particular care over the coefficients. This was where their slip occurred. When reviewing answers, this could have been part of a checklist of common errors to look out for.

Example Candidate Response – low

7 The variables x and t satisfy the differential equation

$$e^{3t}\frac{dx}{dt} = \cos^2 2x,$$

for $t \geqslant 0$. It is given that $x = 0$ when $t = 0$.

(a) Solve the differential equation and obtain an expression for x in terms of t. [7]

$e^{3t}\dfrac{dx}{dt} = \cos^2 2x \, dt$

$\Rightarrow \displaystyle\int \frac{1}{\cos^2 2x}\,dx = \int \frac{1}{e^{3t}}\,dt$ 1

2 $\Rightarrow 2\displaystyle\int \frac{1}{1+\cos 4x}\,dx = \int e^{-3t}\,dt$

3

$\dfrac{2}{x + \frac{1}{4}\sin 4x} = \dfrac{e^{-3t}}{-3} + c$

$2(x + \frac{1}{4}\sin 4x)^{-1} = -\frac{1}{3}e^{-3t} + c$ 4

When $x = 0$, $t = 0$

$\Rightarrow 2(0 + \frac{1}{4}\sin 4(0))^{-1} = -\frac{1}{3}e^{-3(0)} + c$

$0 = -\frac{1}{3} + c$

$c = \frac{1}{3}$

$\Rightarrow \dfrac{2}{x + \frac{1}{4}\sin 4x} = -\frac{1}{3}e^{-3t} + \frac{1}{3}$

Examiner comments

1 The candidate separates the variables correctly.

2 The candidate does not recognise the trig integral as a standard form and attempts to integrate using an incorrect method.

3 The candidate integrates the exponential function correctly.

4 The candidate follows the correct process for finding the constant of integration, but to be awarded the method mark they would need to be working with functions of the correct form.

Example Candidate Response – low, continued	Examiner comments
$x + \frac{1}{4}\sin 4x = -6e^{-3t} + 6$ $x + \frac{1}{4}\sin 4x = -6(e^{-3t} - 1)$ 5 $x^{\times 4} + \frac{1}{4}\sin 4x = -6(e^{-3t} - 1)^{\times 4}$ $4x + \sin 4x = -24(e^{-3t} - 1)$	5 Because the candidate does not obtain the method mark, they are awarded no further marks. Mark for (a) = 2 out of 7
(b) State what happens to the value of x when t tends to infinity. [1] x increases 6	6 To be awarded this mark, the candidate needs to have an expression in part **(a)** whose limiting behaviour could be considered as t tends to infinity. A precise statement was needed. Mark for (b) = 0 out of 1 **Total mark awarded = 2 out of 8**

How the candidate could have improved their answer

The candidate recognised that they needed to deal with $\frac{1}{\cos^2 2x}$. They used a correct trig substitution, but it was not helpful here because they should have known that $\int \frac{1}{f(x)}dx \neq \frac{1}{\int f(x)dx}$. The integral required was a simple variant of a form given in the formula booklet. The candidate needed to prepare thoroughly to help them recognise standard forms, and to make effective use of the resources available to them in the exam.

Common mistakes candidates made in this question

- In part **(a)**, incorrect separation of the variables.
- In part **(a)**, errors in the coefficients when integrating $\sec^2 2x$ and e^{-3t}.
- In part **(a)**, integrating e^{-3t} to obtain ke^{-2t}.
- In part **(a)**, not recognising $\cos^{-2} 2x$ as $\sec^2 2x$, and employing various alternative incorrect methods to integrate the function.
- In part **(a)**, errors in rearranging the solution to obtain an expression for x.
- In part **(b)**, not stating a value for the limit.
- In part **(b)**, stating the value of $\frac{1}{2}\tan^{-1}\left(\frac{2}{3}\right)$ in degrees.

Question 8

Example Candidate Response – high	Examiner comments

8 With respect to the origin O, the position vectors of the points A, B, C and D are given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OD} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}.$$

(a) Show that $AB = 2CD$. [3]

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{AB} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix}$ $\quad \overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$\Rightarrow \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$AB = 2CD.$ (1)

$\begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ $\therefore$ AB = 2CD is equal (Parralel) but in different directions. (2)

$\begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$ (3)

(b) Find the angle between the directions of $\overrightarrow{AB}$ and $\overrightarrow{CD}$. [3]

$\cos\theta = \dfrac{AB \cdot CD}{|AB| \cdot |CD|}$ $\quad |AB| = \sqrt{2^2 + 2^2 + 4^2}$ $= 2\sqrt{6}$ units.

$\cos\theta = \dfrac{\begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}{(2\sqrt{6}) \cdot (\sqrt{6})}$ $\quad |CD| = \sqrt{2^2 + 1^2 + 1^2}$ $= \sqrt{6}$ units

$= \dfrac{(2 \times 2) + (-2 \times 1) + (-4 \times 1)}{12}.$

$\cos\theta = \dfrac{4 - 2 - 4}{12}$ (4)

$\cos\theta = -1/6$ (5)

$\theta = \cos^{-1}(-1/6) = 80.4°$

Examiner comments

(1) The candidate states correct values for both vectors.

(2) This is an incorrect statement about the two vectors.

(3) The candidate makes no attempt to find the lengths of the vectors, so no further marks are available.
Mark for (a) = 1 out of 3

(4) The candidate calculates the scalar product correctly.

(5) The candidate completes the process for finding the angle, but they disregard the negative value for cosine of the angle and give an incorrect answer.
Mark for (b) = 2 out of 3

Example Candidate Response – high, continued

(c) Show that the line through A and B does not intersect the line through C and D. [4]

Line of AB $= \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix}$

Parametric Form

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2+2\lambda \\ 1-2\lambda \\ 5-4\lambda \end{pmatrix} \begin{matrix} \text{—(i)} \\ \text{—(ii)} \\ \text{—(iii)} \end{matrix}$

Line of CD $= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Parametric form.

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+2\mu \\ 1+\mu \\ 2+\mu \end{pmatrix} \begin{matrix} \text{—(i)} \\ \text{—(ii)} \\ \text{—(iii)} \end{matrix}$ 6

$2+2\lambda = 1+2\mu \Rightarrow 2\lambda - 2\mu = -1$ — (i).

$1-2\lambda = 1+\mu \Rightarrow 2\lambda + \mu = 0$ — (ii)

$5-4\lambda = 2+\mu \Rightarrow 4\lambda + \mu = 3$ — (iii)

Simultaneously solve Eq (i) and Eq (ii).

$\mu = -2\lambda$

$2\lambda - 2(-2\lambda) = -1 \Rightarrow 2\lambda + 4\lambda$ 7 $= -1$

$6\lambda = -1 \quad \lambda = -1/6.$

$\mu = -2\left(-\frac{1}{6}\right) = \frac{1}{3}.$

Put values of 'λ' and 'μ' in eq (iii)

$4\left(-\frac{1}{6}\right) + \frac{1}{3} = 3$

$-\frac{2}{3} + \frac{1}{3} = 3 \Rightarrow \boxed{-\frac{1}{3} = 3}$ 8

so Line AB does not intersect Line CD. Showed

Examiner comments

6 The candidate states the equations of both lines correctly.

7 The candidate uses a correct method to find the point of intersection of the two lines.

8 The candidate completes their argument correctly.
Mark for (c) = 4 out of 4

Total mark awarded = 7 out of 10

How the candidate could have improved their answer

- The candidate needed to be familiar with the notation for the exam as set out in the syllabus. In part **(a)**, the candidate confused AB with $\overrightarrow{AB}$ and, consequently, they have not done what the question asked them to do.
- In part **(b)**, the working was correct, but the candidate has not taken account of the negative value for $\cos\theta$, something that they might have picked up through careful checking.
- The solution to part **(c)** was fully correct.

Example Candidate Response – middle	Examiner comments
8 With respect to the origin O, the position vectors of the points A, B, C and D are given by $$\overrightarrow{OA}=\begin{pmatrix}2\\1\\5\end{pmatrix},\quad \overrightarrow{OB}=\begin{pmatrix}4\\-1\\1\end{pmatrix},\quad \overrightarrow{OC}=\begin{pmatrix}1\\1\\2\end{pmatrix}\quad\text{and}\quad \overrightarrow{OD}=\begin{pmatrix}3\\2\\3\end{pmatrix}.$$	
(a) Show that $AB = 2CD$. [3] $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ $\overrightarrow{AB} = \begin{pmatrix}4\\-1\\1\end{pmatrix} - \begin{pmatrix}2\\1\\5\end{pmatrix} = \begin{pmatrix}2\\-2\\-4\end{pmatrix}$ $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$ $\overrightarrow{CD} = \begin{pmatrix}3\\2\\3\end{pmatrix} - \begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix}$ 1	1 The candidate states the two vectors correctly.
$\overrightarrow{AB} = k\,\overrightarrow{CD}$ $\begin{pmatrix}2\\-2\\-4\end{pmatrix} = 2\begin{pmatrix}2\\1\\1\end{pmatrix}$ 2	2 There is no attempt to find the length (modulus) of either vector, so no further marks are awarded. Mark for (a) = 1 out of 3
(b) Find the angle between the directions of $\overrightarrow{AB}$ and $\overrightarrow{CD}$. [3] $\overrightarrow{AB}\cdot\overrightarrow{CD} = \|AB\|\|CD\|\cos\theta$ $\overrightarrow{AB}\cdot\overrightarrow{CD}$:- $(2\times2)+(-2\times1)+(-4\times1) = \overrightarrow{AB}\cdot\overrightarrow{CD}$ $-2 = \overrightarrow{AB}\cdot\overrightarrow{CD}$ 3	3 The candidate uses the correct vectors and the correct method for finding the scalar product.
$\|AB\| = \sqrt{2^2+(-2)^2+(-4)^2}$ $\|AB\| = 2\sqrt{6}$ $\|CD\| = \sqrt{2^2+1^2+1^2}$ $\|CD\| = \sqrt{6}$ $\dfrac{-2}{(2\sqrt{6})(\sqrt{6})} = \cos\theta$ 4	4 The candidate uses the correct method to complete the process to obtain the required angle.
$\boxed{80.4^\circ = \theta}$ 5	5 Despite obtaining a negative value for the cosine of the angle, the candidate gives an acute answer. Mark for (b) = 2 out of 3

Example Candidate Response – middle, continued

(c) Show that the line through A and B does not intersect the line through C and D. [4]

Line passing through AB:-

$$r = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$$ 6

Line passing through CD:-

$$r = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2+2\mu \\ 1-2\mu \\ 5+4\mu \end{pmatrix} = \begin{pmatrix} 1+2\lambda \\ 1+\lambda \\ 2+\lambda \end{pmatrix}$$

7

$2+2\mu = 1+2\lambda$

$1+2\mu = 2\lambda$ — (i)

$1-2\mu = 1+\lambda$

$-2\mu = \lambda$ — (ii)

Substituting λ from eq (ii) in eq (i).

$1+2\mu = 2(-2\mu)$

$1+2\mu = -4\mu$

$1 = -6\mu$

$\boxed{-\frac{1}{6} = \mu}$

$-2\left(-\frac{1}{6}\right) = \lambda$

$\boxed{\frac{1}{3} = \lambda}$

Putting λ & μ in e

$5+4\left(-\frac{1}{6}\right) = 2+\frac{1}{3}$

$\frac{13}{3} \neq \frac{7}{3}$ 8

Hence lines do no intersect.

Examiner comments

6 The candidate miscopies their answer for the direction of AB.

7 The candidate sets up the equations of the two lines correctly and uses the correct method to find the point of intersection.

8 The candidate reaches the required conclusion, but this is fortuitous due to the earlier accuracy error. Mark for (c) = 1 out of 4

Total mark awarded = 5 out of 10

How the candidate could have improved their answer

- The response to part **(b)** confirms that the candidate knew how to find the magnitude of a vector, but they did not recognise the notation for this in part **(a)**. They needed to be familiar with the notations listed in the specification.
- The candidate could have read the question more carefully. Part **(b)** did not ask for the acute angle between two vectors, so the negative value for $\cos\theta$ needed to be considered.
- In part **(c)**, the candidate had miscopied their own writing from part **(a)**. As a result, the method mark was available but not the accuracy marks. They could have checked that they had transferred figures correctly in their working.

Example Candidate Response – low

8 With respect to the origin O, the position vectors of the points A, B, C and D are given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OD} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}.$$

(a) Show that $AB = 2CD$. [3]

$AB = OB - OA$

$= \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix}$ 1

$CO = OD - OA$ 2

$= \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$

$= 2\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}$

3

(b) Find the angle between the directions of $\overrightarrow{AB}$ and $\overrightarrow{CD}$. [3]

$\begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \sqrt{2^2 + (-2)^2 + (-4)^2} \cdot \sqrt{1^2 + 1^2 + 2^2} \cdot \cos\theta$

4 $-8 = 2\sqrt{6} \cdot \sqrt{6} \cos\theta$

$\cos\theta = \left(\dfrac{-8}{2\sqrt{6} \times \sqrt{6}} \right)$

$\theta = \cos^{-1}\left(\dfrac{-8}{2\sqrt{6} \times \sqrt{6}} \right)$

$= 131.8^\circ$ 5

Examiner comments

1 This vector is correct.

2 The candidate uses OA not OC here. To be awarded the first mark, they need both correct vectors.

3 The candidate does not consider the lengths of their vectors, so no further marks are available in part **(a)**.
Mark for (a) = 0 out of 3

4 The candidate makes a sign slip here in copying their vector CD. They are awarded the method mark for correct evaluation of the scalar product.

5 The candidate correctly completes the process for finding the angle, but the earlier error means that they do not obtain the correct answer.
Mark for (b) = 2 out of 3

Example Candidate Response – low, continued

(c) Show that the line through A and B does not intersect the line through C and D. [4]

6 $\begin{pmatrix}2\\-2\\-4\end{pmatrix} + \lambda\begin{pmatrix}4\\-1\\1\end{pmatrix}$ $\qquad$ $\begin{pmatrix}1\\1\\-2\end{pmatrix} + \mu\begin{pmatrix}3\\2\\3\end{pmatrix}$

7

$$\begin{pmatrix}2+4\lambda\\-2-\lambda\\-4+\lambda\end{pmatrix} \qquad \begin{pmatrix}1+3\mu\\1+2\mu\\-2+3\mu\end{pmatrix}$$ 8

$2+4\lambda = 1+3\mu \quad \ldots \; ①$

$-2-\lambda = 1+2\mu \quad \ldots \; ②$

$-4+\lambda = -2+3\mu \quad \ldots \; ③$

$4\lambda - 3\mu = -1$

$\lambda + 2\mu = -3$

9

$\lambda = -3-2\mu$

$4(-3-2\mu)-3\mu = -1 \qquad \lambda = -3-2(-1)$

$-12-8\mu-3\mu = -1 \qquad \lambda = -1$

$-11\mu = 11$

$\mu = -1$

10

$-4+\lambda = -2+3\mu$

$-4-1 = -2+3(-1)$

$-5 =$

Examiner comments

6 This is the candidate's direction vector.

7 This is a position vector of a point on the line.

8 The candidate has not set up the equation of the line correctly.

9 The candidate applies the correct method for finding the point of intersection of two lines.

10 Although the candidate uses the correct process, incorrect equations result in an incorrect answer.
Mark for (c) = 1 out of 4

Total mark awarded = 3 out of 10

How the candidate could have improved their answer

- The candidate needed to understand the notation used in part **(a)**.
- In **(b)**, the candidate was given the benefit of doubt over the error in the vector, and was allowed credit for correct use of the method to find an angle.
- In part **(c)**, the candidate had confused position vectors and direction vectors, so the only mark available was the method mark for finding out where their lines meet.
- The candidate could have checked for errors in the working. This response showed knowledge of the correct processes, but more precision was needed in applying them.

Common mistakes candidates made in this question

- In part **(a)**, sign slips in stating the required vectors.
- In part **(a)**, not using the modulus to compare the magnitudes of the two vectors.
- In part **(b)**, not using the correct vectors.
- In part **(b)**, sign slips in copying their vectors.
- In part **(b)**, stating the final answer as an acute angle.
- In part **(c)**, sign slips in copying the vectors from earlier working.
- In part **(c)**, confusion between directions and position vectors when setting up the equations of the two lines.
- In part **(c)**, using the same parameter in both line equations, making it impossible to solve for the point of intersection.
- In part **(c)**, errors in solving the equations.
- In part **(c)**, no concluding statement, e.g., some candidates reached a statement such as $\frac{17}{3} = \frac{7}{3}$ but wrote nothing more.

Question 9

Example Candidate Response – high

9 Let $\mathrm{f}(x) = \dfrac{7x+18}{(3x+2)(x^2+4)}$.

(a) Express $\mathrm{f}(x)$ in partial fractions. [5]

$$\frac{7x+18}{(3x+2)(x^2+4)} = \frac{A}{(3x+2)} + \frac{Bx+C}{(x^2+4)}$$ 1

$$7x+18 = A(x^2+4) + Bx(3x+2) + C(3x+2)$$

$$3x+2=0 \qquad x=-\frac{2}{3}$$

$$7\left(-\frac{2}{3}\right)+18 = A\left[\left(-\frac{2}{3}\right)^2+4\right]$$

$$A=3$$ 2

$$7(0)+18 = 3(4) + C(2)$$

$$C = \cancel{2}\ 5$$ 3

$$7(1)+18 = 3(5)+B(5)+5(5)$$

$$B=-3$$ 4

$$\therefore \frac{3}{(3x+2)} + \frac{5-3x}{(x^2+4)}$$

(b) Hence find the exact value of $\displaystyle\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$. [6]

$$\int_0^2 \frac{3}{(3x+2)} + \frac{5}{(x^2+4)} - \frac{3x}{(x^2+4)}$$ 5 6

$$\left[\frac{3\ln(3x+2)}{3} + \frac{5\ln(x^2+4)}{2} - \frac{3\ln(x^2+4)}{2}\right]_0^2$$ 7 8

$$\left[\ln(6+2) + \frac{5}{2}\ln(8) - \frac{3}{2}\ln(8)\right] - \left[\ln 2 + \frac{5}{2}\ln 4 - \frac{3}{2}\ln 4\right]$$

$$\left[\ln 8 + \ln 8^{\frac{5}{2}} - \cancel{\ln 8}\ \ln 8^{\frac{3}{2}}\right] - \left[\ln 2 + \ln 4^{\frac{5}{2}} - \ln 4^{\frac{3}{2}}\right]$$

$$\left[\ln \frac{8 \times 8^{\frac{5}{2}}}{8^{\frac{3}{2}}}\right] - \cancel{\ln}\left[\ln \frac{2\times 4^{\frac{5}{2}}}{4^{\frac{3}{2}}}\right]$$ 9

$$\ln 64 - \ln 8$$

$$\ln \frac{64}{8}$$

$$= \ln 8$$

Examiner comments

1 The candidate selects the correct form for the partial fractions.

2 The candidate obtains the correct value for A.

3 The candidate has the correct equation for C but makes an error in solving it.

4 An incorrect value for B follows from the earlier error.
Mark for (a) = 3 out of 5

5 The candidate integrates the first term correctly.

6 The candidate recognises the need to split the second fraction into two separate terms.

7 The candidate does not recognise the form of the second integral.

8 The third integral follows correctly from the candidate's coefficients.

9 Two of the candidate's three terms are of the correct form. They use the limits correctly, but their final answer is not correct due to the earlier errors.
Mark for (b) = 4 out of 6

Total mark awarded = 7 out of 11

How the candidate could have improved their answer

- The error in the arithmetic in part **(a)** could have been detected through careful checking of the working.
- The accuracy errors in part **(a)** affected the answer in **(b)**, but a more important issue here was that the candidate did not recognise a standard integral. A quick check would have confirmed that this term was not of the form $\frac{f'(x)}{f(x)}$.

Example Candidate Response – middle

9 Let $f(x) = \dfrac{7x + 18}{(3x + 2)(x^2 + 4)}$.

(a) Express $f(x)$ in partial fractions. [5]

1

$7x + 18 = A(x^2 + 4) + Bx + C(3x + 2)$

x as $-2/3$

$13\frac{1}{3} = 4\frac{4}{9}A$

$3 = A$ 2

x as 0

$18 = 12 + 2C$

$6 = 2C$

$3 = C$ 3

x as 1

$25 = 15 + B + 15$

$-5 = B$ 4

$\therefore f(x) = \dfrac{3}{3x+2} - \dfrac{5x+3}{x^2+4}$

(b) Hence find the exact value of $\int_0^2 f(x)\,dx$. [6]

$\int_0^2 \dfrac{3}{3x+2} + (-5x+3)(x^2+4)^{-1}$

Integrating $(-5x+3)(x^2+4)^{-1}$

$\cancel{u = -5x+3}$ $\quad \dfrac{du}{dx} = -5$

$u = (x^2+4)^{-1} \quad \dfrac{du}{dx} = -1(x^2+4)^{-2}$

$\dfrac{dv}{dx} = -5x+3 \quad v = -\dfrac{5}{2}x^2 + 3x$

Integrating $\dfrac{3}{3x+2}$

$= \ln(3x+2)$

$[\ln(3x+2)$

$\int_0^2 \dfrac{3}{3x+2} - 5x + 3\left(\dfrac{1}{x^2+4}\right) dx$ 6

5 $\left[\ln(3x+2) - 5x + 3\left(\dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right)\right)\right]_0^2$

$\left[\ln 3x + \ln 2 - \dfrac{5}{2}x\tan^{-1}\left(\dfrac{x}{2}\right) + \dfrac{3}{2}\tan^{-1}\dfrac{x}{2}\right]_0^2$ 7

$\left(\ln 6 + \ln 2 - 5x\tan^{-1}\left(\dfrac{2}{2}\right) + 3\tan^{-1}\dfrac{x2}{2}\right) - (\ln 2)$

$\dfrac{\ln 12 - 2\left(\frac{1}{4}\pi\right)}{\ln 2}$ 8

$= \dfrac{\ln 12 - \frac{\pi}{2}}{\ln 2}$

Examiner comments

1 This equation implies that the candidate is using the correct form for the partial fractions, but for it to be correct, they should have brackets around $Bx + C$. At this stage, it is not possible to tell whether or not the candidate knows that the brackets should be there.

2 The candidate uses a correct method to find the value of A and their answer suggests that they are aware of the 'invisible brackets' in their initial equation.

3 The candidate obtains the correct value for C.

4 The equation for B is not using the 'invisible brackets' and consequently the candidate obtains an incorrect value.
Mark for (a) = 4 out of 5

5 The candidate integrates the first term correctly.

6 The candidate is not using the brackets around the numerator in the first line, but their intention becomes clear on the third line.

7 The candidate is not using a correct method to integrate the second term.

8 Although the candidate appears to have two terms of the correct form, these have not been obtained correctly, so the method mark for correct use of the limits is not available.
Mark for (b) = 1 out of 6

Total mark awarded = 5 out of 11

How the candidate could have improved their answer

- Using brackets in the initial working for part **(a)** would have improved the candidate's chances of reaching a correct answer.
- In **(b)**, the error arose in treating $5x+3$ as a scalar multiple of the integral, not as a function of x. When starting work on the integral, the candidate could have looked to see if it could be split into terms that follow a known pattern, i.e. not by treating a variable as a constant.

Example Candidate Response – low

9 Let $f(x) = \dfrac{7x+18}{(3x+2)(x^2+4)}$.

(a) Express $f(x)$ in partial fractions. [5]

$$\frac{7x+18}{(3x+2)(x^2+4)} = \frac{A}{(3x+2)} + \frac{Bx+C}{(x^2+4)}$$ 1

$$\cancel{7x+18} = \frac{A}{\cancel{(3x+2)}} + \frac{\cancel{Bx+C}}{\cancel{(x^2+4)}}$$

$$7x+18 = (x^2+4)A + (Bx+C)(3x+2)$$

$-\frac{2}{3}$

$x = -\frac{2}{3}$

$$7\left(-\frac{2}{3}\right)+18 = \left(\left(-\frac{2}{3}\right)^2+4\right)A$$

$$\frac{40}{3} = \frac{40}{9}A$$ 2

$A = 3$

$x = 0$

$\cancel{= 4B + 2C}$

$18 = 2C$ 3

$C = 9$

$x = 1$

$$7+18 = (5\times 3) + (B+9)(5)$$

$\cancel{5B+45}$

$25 - 15 = +5B + 45$

$-35 = 5B$

$B = -7$

$$\frac{7x+18}{(3x+2)(x^2+4)} = \frac{3}{(3x+2)} + \frac{-7x+9}{(x^2+4}$$

(b) Hence find the exact value of $\int_0^2 f(x)\,dx$. [6]

$$\int \frac{3}{(3x+2)} + \int \frac{-7x+9}{(x^2+4)} \cancel{+c}$$

$$\left[\cancel{3}\,\frac{\ln 3x+2}{\cancel{3}} + \left(\frac{-7x^2}{2}+9x\right)\frac{\ln(x^2+4)}{2x}\right]_0^2$$ 4 5

$x=2$

$$\ln(3\times 2)+2 + \left[\frac{-7(2)^2}{2}+9(2)\right]\times\frac{\ln(2^2+4)}{4}$$

$x=0$

$$\ln 0+2 + 0\times\frac{\ln(0+4)}{0}$$ 6

$\cancel{3+19}$ $\quad 4\times\frac{\ln 8}{4} + \ln 6 + 2 - 0$

$\ln 8 + \ln 6 + 2$

5.871201011

Examiner comments

1 The candidate selects the correct form for the partial fractions.

2 The candidate uses a correct method to obtain the correct value for A.

3 The candidate makes an error in substituting $x = 0$, leading to an incorrect answer for C and subsequently an incorrect answer for B.
Mark for (a) = 3 out of 5

4 The candidate integrates the first fraction correctly.

5 The candidate has not recognised the need to split this fraction into two parts before it can be integrated.

6 The candidate attempts to use the limits but the method mark is not awarded because they are not working with terms of the correct form.
Mark for (b) = 1 out of 6

Total mark awarded = 4 out of 11

How the candidate could have improved their answer

- The candidate could have noticed the error in part **(a)** through checking.
- In part **(b),** the first step should have been to split the second fraction into terms that followed recognised forms rather than using this relationship which is not true: $\int \mathrm{f}(x)\,\mathrm{g}(x)\mathrm{d}x = \int \mathrm{f}(x)\mathrm{d}x \times \int \mathrm{g}(x)\mathrm{d}x$.

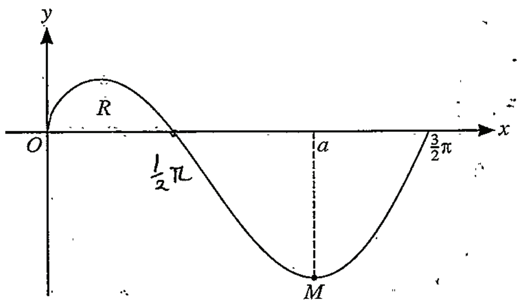
Common mistakes candidates made in this question

- In part **(a)**, use of the incorrect form $\frac{A}{3x+2} + \frac{B}{x^2+4}$.
- In part **(a)**, rewriting x^2+4 as $(x+2)^2$ or as $(x-2)(x+2)$.
- In part **(a)**, the use of 8 in place of 18 in the course of their working.
- In part **(a)**, slips in solving correct equations.
- In part **(b)**, errors in the coefficient when integrating $\frac{A}{3x+2}$.
- In part **(b)**, attempting to integrate $\frac{Bx+C}{x^2+4}$ as a single term.
- In part **(b)**, errors in simplifying after the correct use of limits.

Question 10

Example Candidate Response – high	Examiner comments

10



The diagram shows the curve $y = \sqrt{x}\cos x$, for $0 \leqslant x \leqslant \frac{3}{2}\pi$, and its minimum point M, where $x = a$. The shaded region between the curve and the x-axis is denoted by R.

(a) Show that a satisfies the equation $\tan a = \frac{1}{2a}$. [3]

$x^{\frac{1}{2}}\cos x$

$\frac{dy}{dx}$ $\frac{1}{2}x^{-\frac{1}{2}}\cos x - x^{\frac{1}{2}}\sin x = 0$ 1

$\frac{1}{2}x^{-\frac{1}{2}}\cos x = x^{\frac{1}{2}}\sin x$

$\frac{1}{2x^{\frac{1}{2}}} \cdot \frac{1}{x^{\frac{1}{2}}} = \tan x$

$\frac{1}{2x} = \tan x$ @ $x = a$ 2

$\frac{1}{2a} = \tan(a)$ Proved.

1 The candidate differentiates correctly using the product rule.

2 The candidate rearranges their answer and shows sufficient working to confirm that they have obtained the given answer correctly.
Mark for (a) = 3 out of 3

(b) The sequence of values given by the iterative formula $a_{n+1} = \pi + \tan^{-1}\left(\frac{1}{2a_n}\right)$, with initial value $x_1 = 3$, converges to a.

Use this formula to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_1 = 12.6039$	$x_8 = 7.0829$	7.1458
$x_2 = \cancel{12}\ 5.413$	$x_9 = 7.1795$	7.1441
$x_3 = 8.4187$	$x_{10} = 7.1254$	7.1451
$x_4 = 6.5405$	$x_{11} = 7.1555$	7.1445
$x_5 = 7.5132$	$x_{12} = 7.1389$ 3	
$x_6 = 6.949$	$x_{13} = 7.1481$	
$x_7 = 7.257$	7 7.1428	Answer = 7.14

3 The candidate appears to understand the method for using an iterative formula, but they are working in degrees so no marks are available.
Mark for (b) = 0 out of 3

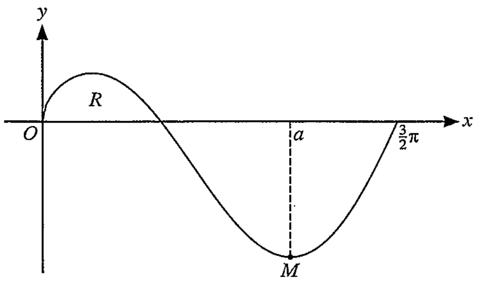
Example Candidate Response – high, continued	Examiner comments
(c) Find the volume of the solid obtained when the region R is rotated completely about the x-axis. Give your answer in terms of π. [6]	
$\cos x = 0 \quad x = \frac{1}{2}\pi$ limits = 0 and $\frac{1}{2}\pi$ $\pi \int (x^{\frac{1}{2}} \cos x)^2$ **4**	**4** The candidate selects the correct method for finding the required volume.
$x \cos^2 x$ $u = x \quad u' = 1$ $v' = \cos^2 x \quad v = \frac{1}{2}\cos 2x + \frac{1}{2}$ **5** $v = \frac{1}{4}\sin 2x + \frac{1}{2}x$	**5** The candidate uses the correct substitution to be able to integrate the trigonometric function.
$\frac{1}{2}x^2 + \frac{1}{4}x\sin 2x - \int \cancel{\frac{1}{2}\cos 2x + \frac{1}{2}}$ $-\int \frac{1}{4}\sin 2x + \frac{1}{2}x$ **6** $-\int -\frac{1}{8}\cos 2x + \frac{1}{4}x^2$	**6** The candidate works through the integration by parts correctly.
$\left[\frac{1}{2}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x - \frac{1}{4}x^2\right]_{\frac{1}{2}\pi}^{0}$ $\frac{1}{4}\pi\left[x\sin 2x + \frac{1}{2}\cos 2x - x^2\right]_{\frac{1}{2}\pi}^{0}$ **7** $\cancel{\frac{1}{8}\pi - \left(\frac{1}{2} + \left(\frac{1}{2}\pi\right)^2\right)}$ $\left(+\frac{1}{2} + \frac{1}{4}\pi^2\right)$ $\frac{1}{8}\pi + \frac{1}{8}\pi = \frac{1}{4}\pi$ $\rightarrow \frac{1}{4}\pi - \frac{1}{16}\pi^3$	**7** The candidate uses the limits the wrong way round, and this leads to the negative of the correct answer. Mark for (c) = 4 out of 6 **Total mark awarded = 7 out of 12**

How the candidate could have improved their answer

- Part **(a)** was fully correct.
- The question made it clear that x was measured in radians, and the candidate should have been aware that when completing an iteration involving a trig function, they should work in radians. No marks were available in part **(b)** for working in degrees.
- In part **(c)**, the candidate's work to find the volume was all correct until they used the limits the wrong way around. They could have asked themselves whether the final answer made sense: the negative value should have provided a hint that there was an error.

Example Candidate Response – middle

10



The diagram shows the curve $y = \sqrt{x}\cos x$, for $0 \leqslant x \leqslant \frac{3}{2}\pi$, and its minimum point M, where $x = a$. The shaded region between the curve and the x-axis is denoted by R.

(a) Show that a satisfies the equation $\tan a = \frac{1}{2a}$. [3]

$\frac{dy}{dx} = x^{\frac{1}{2}}(-\sin x) + \cos x\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$ ~~$y = x^{\frac{1}{2}}\cos x$~~ ~~$\sqrt{a}\cos a$~~ 1

$= -\sqrt{x}\sin x + \frac{\cos x}{2\sqrt{x}}$

$0 = -\sqrt{a}\sin a + \frac{\cos a}{2\sqrt{a}}$

$\sqrt{a}\sin a = \frac{\cos a}{2\sqrt{a}}$

$2a\sin a = \cos a$ 2

$\tan a = \frac{1}{2a}$

$a = a^{\frac{1}{2}}\cos a$

$\frac{a}{\cos a} = \sqrt{a}$

$\frac{a^2}{\cos^2 a} = a$

$\frac{a^2}{}$

(b) The sequence of values given by the iterative formula $a_{n+1} = \pi + \tan^{-1}\left(\frac{1}{2a_n}\right)$, with initial value $x_1 = 3$, converges to a.

Use this formula to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$a_1 = 3$

$a_2 = 3.3067$

$a_3 = 3.2917$ 3

$a_4 = 3.2923$

$a_5 = 3.2923$

$a_6 = 3.2923$

$a_7 = 3.2923$ 4

$a = 3.30$ to 2dp 5

Examiner comments

1 The candidate uses the correct method for differentiation of a product, and they obtain the correct answer.

2 The candidate shows clear working and they obtain the given answer correctly.
Mark for (a) = 3 out of 3

3 The candidate applies the iterative process correctly.

4 The candidate gives sufficient correctly rounded results to support a correct answer.

5 The candidate makes a rounding error in their conclusion.
Mark for (b) = 2 out of 3

Example Candidate Response – middle, continued	Examiner comments

(c) Find the volume of the solid obtained when the region R is rotated completely about the x-axis. Give your answer in terms of π. [6]

$V = \pi \int_a^b y^2 \, dx$ $\qquad \sin 2x = 2\sin x \cos x$

$y = \sqrt{x}\cos x$ $\qquad u^2 \quad \int \frac{u^3}{3}$

$y^2 = (\sqrt{x}\cos x)^2 = x\cos^2 x$

when $y = 0$: $0 = \sqrt{x}\cos x$

$\sqrt{x} = 0$ or $\cos x = 0$

$x = 0$ or $x = \frac{\pi}{2}$

$x\cos^2 x$ $\quad u = \cancel{\cos^2} x$ $\quad \frac{du}{dx} = 1$ $\quad \therefore v = \frac{\cos^3 x}{3}$ $\quad \frac{dv}{dx} = \cos^2 x$

$V = \pi \int_0^{\pi/2} x\cos^2 x \, dx$ 6

$u = \cos^2 x$ $\qquad v = 1$

$\frac{du}{dx} = 2\cos x(-\sin x)$ $\qquad \frac{dv}{dx} = x$

$= (\cos^2 x)(1) - \int -2\cos x \sin x \, dx$

$= \cos^2 x - 1\int 2\cos x \sin x \, du$ 7

$= \cos^2 x - 1 \int \sin 2x \, dx$

$= \cos^2 x - 1\left(-\cos 2x \times \frac{1}{2}\right)$

$= \cos^2 x - \frac{1}{2}\cos 2x$

$V = \pi\left[\cos^2 x - \frac{1}{2}\cos 2x\right]_0^{\pi/2}$

$V = \pi\left(\cos^2\left(\frac{\pi}{2}\right) - \frac{1}{2}\cos\left(2\left(\frac{\pi}{2}\right)\right)\right) - \left(\cos^2(0) - \frac{1}{2}\cos(2\times 0)\right)$

$= \pi\left(0 - \frac{1}{2}(-1)\right) - \left(1 - \frac{1}{2}\right)$

$= \left(\frac{1}{2} + \frac{1}{2}\right)\pi$

$= \frac{\pi}{4}$

Examiner comments

6 The candidate states the correct function to integrate to find the required volume.

7 The candidate is correct in thinking that they need to use integration by parts, but they are not awarded a mark because they do not use a correct method to integrate the trigonometric function.
Mark for (c) = 1 out of 6

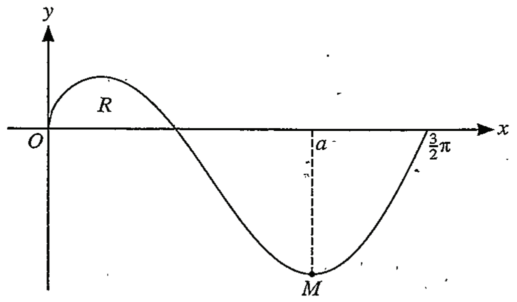
Total mark awarded = 6 out of 12

How the candidate could have improved their answer

- Part **(a)** was fully correct.
- The rounding error in part **(b)** could have been detected if the candidate had time to check their working.
- If the method for integration by parts had been applied correctly in part **(c)**, the candidate would have realised that they had an increasing power of x and so this approach was not leading in a useful direction.

Example Candidate Response – low

10



The diagram shows the curve $y = \sqrt{x}\cos x$, for $0 \leqslant x \leqslant \frac{3}{2}\pi$, and its minimum point M, where $x = a$. The shaded region between the curve and the x-axis is denoted by R.

(a) Show that a satisfies the equation $\tan a = \frac{1}{2a}$. [3]

1

$y = \sqrt{a}\ \sin a$

$y^2 = (\sqrt{a}\ \sin a)^2$

$y^2 = a \times \sin^2 a$

$y^2 = a \times \frac{1}{\operatorname{cosec}^2 a}$

$y = \frac{a}{1+\cot^2 a}$

$y^2 = \frac{a}{1+\frac{1}{\tan^2 a}}$

$1 + \frac{1}{\tan^2 a} = a$

(b) The sequence of values given by the iterative formula $a_{n+1} = \pi + \tan^{-1}\left(\frac{1}{2a_n}\right)$, with initial value $x_1 = 3$, converges to a.

Use this formula to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$a_1 = 3$

$x_{n+1} = x_2 = \pi + \tan^{-1}\left(\frac{1}{2\times 3}\right) = 12.60$

2

$= x_{2+1} = \pi + \tan^{-1}\left(\frac{1}{2\times 12.60}\right) = 5.414.$

$x_{3+1} = \pi + \tan^{-1}\left(\frac{1}{2\times 5.414}\right) = 8.418.$

Examiner comments

1 $\cos a$ has become $\sin a$ but there is no evidence that the candidate has attempted to differentiate a product, so the method mark is not awarded.
Mark for (a) = 0 out of 3

2 The candidate understands the method for applying the iterative process, but they are working in degrees, so no marks are awarded.
Mark for (b) = 0 out of 3

Example Candidate Response – low, continued	Examiner comments
(c) Find the volume of the solid obtained when the region R is rotated completely about the x-axis. Give your answer in terms of π. [6] Volume $= \int_0 \pi y^2 \, dx$. $y = \sqrt{x} \cos x$ put $y = 0$ $0 = \sqrt{x} \cos x$ $0 = x \cos^2 x$. $\pi \int_0 x \cos^2 x$. **3** Applying integration by parts: $u = x \quad \frac{dv}{dx} = \cos^2 x$ $x \times \frac{\sin^2 x}{2} - \int_0 \frac{1}{2} \sin^2 x \times 1$ **4** $\frac{x}{2} \sin^2 x - \frac{1}{2} \int \sin^2 x$ $\frac{x}{2} \sin^2 x - \frac{1}{2} \left[\frac{-\cos^2 x}{2} \right]$ $\frac{x}{2} \sin^2 x + \frac{1}{4} \cos^2 x$	**3** The candidate indicates the correct strategy for finding the volume of the solid. Their notation is not correct: the integral should have ‘dx’ at the end to indicate that the integration is with respect to x. **4** The candidate attempts to use integration by parts, which is the method required here, but they do not have a correct method for integrating the trig function, so no marks are awarded. Mark for (c) = 1 out of 6 **Total mark awarded = 1 out of 12**

How the candidate could have improved their answer

- The hint was in the rubric: M was the minimum point, so differentiating the given function needed to be their first step.
- In using an iterative formula with a trig function, the working needed to use radians, not degrees.
- This candidate made a correct start but, to avoid their error in the integration, they needed to be aware of one of the basic forms: how to complete $\int \cos^2 x \, dx$.

Common mistakes candidates made in this question

- In part **(a)**, slips in the differentiation.
- In part **(a)**, no working to show the rearrangement of the derivative to obtain the given result.
- In part **(b)**, working in degrees.
- In part **(b)**, using an incorrect formula, e.g., tan in place of $\tan^{-1}$.
- In part **(b)**, not working to the accuracy specified in the question.
- In part **(c)**, attempting to use $\int y \, dx$ in place of $\int \pi y^2 \, dx$.
- In part **(c)**, use of an incorrect method to integrate $\cos^2 x$.
- In part **(c)**, use of incorrect limits (usually 0 and $\frac{3}{2}\pi$) to evaluate the integral.

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