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Updated to the 2020-22 Syllabus

CIE A-LEVEL MATHS 9709 (M)

FORMULAE AND SOLVED QUESTIONS FOR MECHANICS (M)

TABLE OF CONTENTS

1. Velocity and Acceleration

1.1 Kinematics Equations

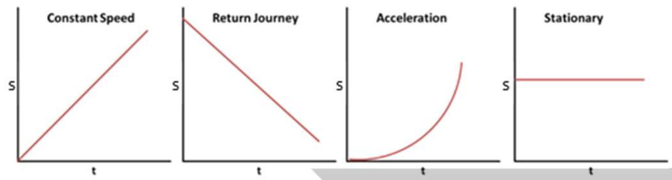
$$v = u + at$$

$$s = ut + \frac{1}{2}at^2 \text{ and } s = vt - \frac{1}{2}at^2$$

$$s = \frac{1}{2}(u+v)t$$

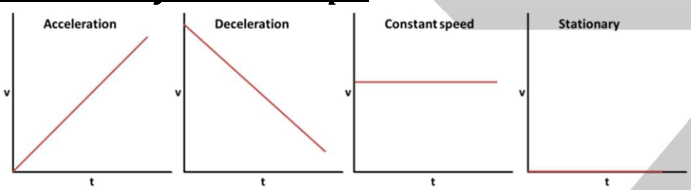
$$v^2 = u^2 + 2as$$

1.2 Displacement-Time Graph



- Gradient = speed

1.3 Velocity-Time Graph



- Gradient = acceleration
- Area under graph = change in displacement

{S12-P42} **Question 7:**

The small block has mass $0.15kg$. The surface is horizontal. The frictional force acting on it is $0.12N$. Block set in motion from X with speed $3ms^{-1}$. It hits vertical surface at Y $2s$ later. Block rebounds from wall directly towards X and stops at Z. The instant that block hits wall it loses $0.072J$ of its kinetic energy. The velocity of the block from X to Y direction is $v\ ms^{-1}$ at time $t\ s$ after it leaves X.

i. Find values of v when the block arrives at Y and when it leaves Y. Also find t when block comes to rest at Z. Then sketch a velocity-time graph of the motion of the small block.

ii. Displacement of block from X, in the $\overrightarrow{XY}$ direction is $s\ m$ at time $t\ s$. Sketch a displacement-time graph. On graph show values of s and t when block at Y and when it comes to rest at Z.

Solution:

Part (i)

Calculating deceleration using Newton's second law:

$$0.12 = 0.15a \qquad a = \frac{0.12}{0.15} = 0.8ms^{-2}$$

Calculate v at Y using relevant kinematics equation

$$-0.8 = \frac{v-3}{2} \qquad v = 1.4ms^{-1}$$

Calculate kinetic energy at Y

$$E_K = \frac{1}{2}(0.15)(1.4)^2 = 0.147J$$

Calculate energy lost:

$$Initial - Change = Final$$

$$0.147 - 0.072 = 0.075J$$

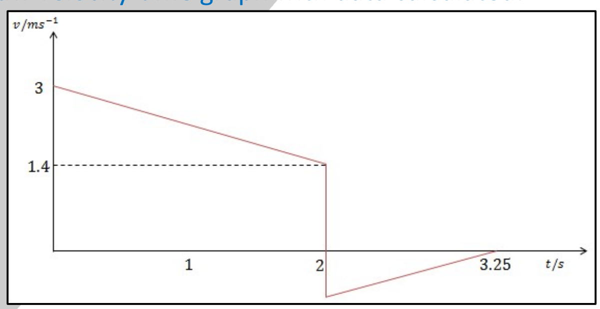
Calculate speed as leaving Y using $k.E.$ formula:

$$0.075 = \frac{1}{2}(0.15)v^2 \qquad v = 1ms^{-1}$$

Calculate t when particle comes to rest:

$$-0.8 = \frac{0-1}{t} \qquad t = 1.25s$$

Draw velocity-time graph with data calculated:



Part (ii)

Calculate displacement from X to Y

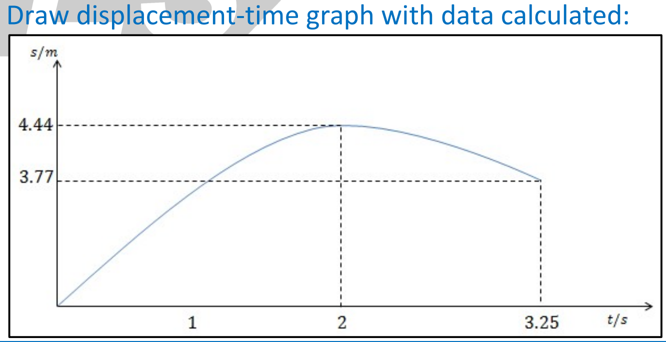
$$s = (3 \times 2) + \frac{1}{2}(-0.8)(2)^2 \qquad s = 4.4m$$

Calculate displacement from Y to Z

$$s = (1 \times 1.25) + \frac{1}{2}(-0.8)(1.25)^2$$

$$s = 0.625m \text{ in the opposite direction}$$

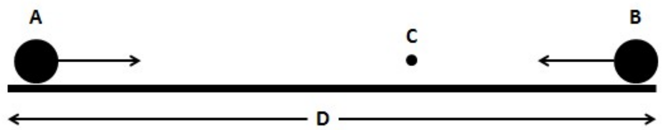
Draw displacement-time graph with data calculated:



1.4 Average Velocity

- For an object moving with constant acceleration over a period of time, these quantities are equal:
  - The average velocity
  - The mean of initial & final velocities
  - Velocity when half the time has passed

1.5 Relative Velocities



- Let s_A be the distance travelled by A and s_B for B

$$s_A = \mathrm{ut} + \frac{1}{2}at^2 \qquad s_B = \mathrm{ut} + \frac{1}{2}at^2$$

- If a collision occurs at point C

$$s_A + s_B = D$$

- This gives you the time of when the collision occurred
- Same analysis if motion is vertical

2. Force and Motion

Newton's 1st Law of Motion:

Object remains at rest or moves with constant velocity unless an external force is applied

Newton's 2nd Law of Motion:

$$F = ma$$

3. Vertical Motion

- **Weight:** directly downwards
- **Normal contact force:** perpendicular to place of contact

3.1 Common Results of Vertical Motion

Finding time taken to reach maximum height by a projectile travelling in vertical motion:

$$v = u + at$$

- Let $v = 0$ and find t
- The time taken to go up and come back to original position would be double of this t

Finding maximum height above a launch point use:

- $v^2 = u^2 - 2as$
- Let $v = 0$ and find s

Finding time interval for which a particle is above a given height:

- Let the height be H and use
- $s = ut + \frac{1}{2}at^2$
- Let $s = H$
- There will be a quadratic equation in t
- Solve and find the difference between the 2 t's to find the time interval

{S04-P04} **Question 7:**

Particle P_1 projected vertically upwards, from horizontal ground, with speed $30ms^{-1}$. At same instant P_2 projected vertically upwards from tower height $25m$, with speed $10ms^{-1}$

i. Find the time for which P_1 is higher than the top of the tower
ii. Find velocities of the particles at instant when they are same height
iii. Find the time for which P_1 is higher than P_2 and moving upwards

Solution:

Part (i)

Substitute given values into displacement equation:

$$25 = (30)t + \frac{1}{2}(10)t^2$$
$$5t^2 + 30t - 25 = 0$$

Solve quadratic for t

$$t = 1s \text{ or } 5s$$

P_1 reaches tower at $t = 1$ then passes it again when coming down at $t = 5s$

Therefore, time above tower $= 5 - 1 = 4$ seconds

Part (ii)

Displacement of P_1 is S_1, and of P_2 is S_2 & relationship:

$$S_1 = 25 + S_2$$

Create equations for S_1 and S_2

$$S_1 = 30t + \frac{1}{2}(-10)t^2 \qquad S_2 = 10t + \frac{1}{2}(-10)t^2$$

Substitute back into initial equation

$$30t + \frac{1}{2}(-10)t^2 = 25 + 10t + \frac{1}{2}(-10)t^2$$

Simple cancelling

$$t = 1.25s$$

Find velocities

$$v = u + at$$
$$V_1 = 30 - 10(1.25) = 17.5ms^{-1}$$
$$V_2 = 10 - 10(1.25) = -2.5ms^{-1}$$

Part (iii)

We know when P_1 and P_2 at same height $t = 1.25s$.

Find time taken to reach max height for P_1

$$v = u + at$$

V is 0 at max height

$$0 = 30 - 10t \qquad t = 3s$$

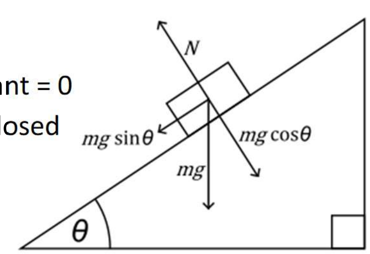
Time for P_1 above $P_2 = 3 - 1.25 = 1.75$ seconds

4. Resolving Forces

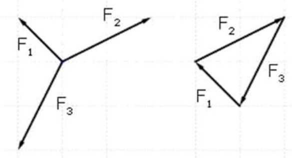
- If force F makes an angle θ with a given direction, the effect of the force in that direction is $F\cos\theta$

$$F\cos(90 - \theta) = F\sin\theta$$
$$F\sin(90 - \theta) = F\cos\theta$$

- Forces in equilibrium: resultant = 0
- If drawn, forces will form a closed polygon



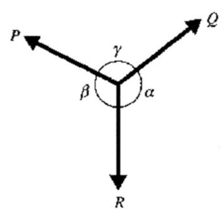
- **Methods of working out forces in equilibrium:**
  - Construct a triangle and work out forces
  - Resolve forces in x and y directions; sum of each = 0



Lami's Theorem:

- For any set of three forces P,Q and R in equilibrium

$$\frac{P}{\sin\theta}=\frac{Q}{\sin\beta}=\frac{R}{\sin\gamma}$$



5. Friction

Friction = Coefficient of Friction × Normal Contact Force

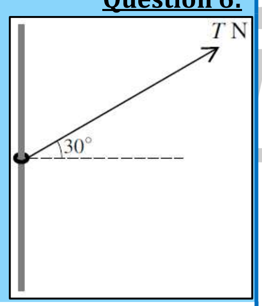
$$F=\mu r$$

- Friction always acts in the opposite direction of motion
- Limiting equilibrium: on the point of moving, friction at max (limiting friction)
- Smooth contact: friction negligible
- **Contact force:**
  - Refers to both F and N
  - Horizontal component of Contact force = F
  - Vertical component of Contact force = N
  - Magnitude of Contact force given by the formula:

$$C=\sqrt{F^2+N^2}$$

{W11-P43} **Question 6:**

The ring has a mass of $2kg$. The horizontal rod is rough and the coefficient of friction between ring and rod is 0.24. Find the two values of T for which the ring is in limiting equilibrium



Solution:

The ring is in limiting equilibrium in two different scenarios; we have to find T in both:

Scenario 1: ring is about to move upwards

$$Resultant=T\sin 30-friction-Weight\ of\ Ring$$

Since the system is in equilibrium, resultant = 0:

$$Contact\ Force=T\cos 30$$
$$\therefore Friction=0.24\times T\cos 30$$

Substitute relevant information in to initial equation

$$0=T\sin 30-0.24T\cos 30-20$$
$$T=68.5N$$

Scenario 2: ring is about to move downwards

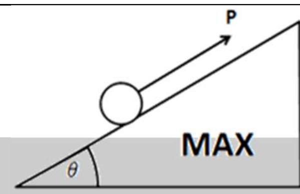
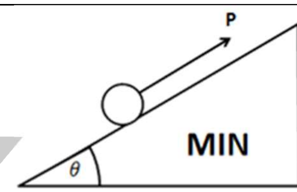
This time friction acts in the opposite direction since friction opposes the direction of motion, thus:

$$Resultant=T\sin 30+Friction-Weight\ of\ Ring$$

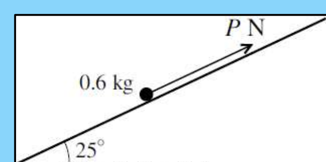
Using information from before:

$$0=T\sin 30+0.24T\cos 30-20$$
$$T=28.3N$$

5.1 Equilibrium

Force required to keep a particle in equilibrium on a rough plane	
Max Value	**Min Value**
 • The particle is about to move up • Thus, friction force acts down the slope $P=F+mg\sin\theta$	 • The particle is about slip down • Thus, frictional force acts up the slope $F+P=mg\sin\theta$

{W12-P43} **Question 6:**



Coefficient of friction is 0.36 and the particle is in equilibrium. Find the possible values of P

Solution:

The magnitude of friction on particle in both scenarios is the same but acting in opposite directions

Calculate the magnitude of friction first:

$$Contact\ Force=6\cos 25$$
$$\therefore Friction=0.36\times 6\cos 25$$

Scenario 1: particle is about to move upwards

$$P=6\sin 25+Friction$$
$$P=4.49N$$

Scenario 2: particle is about to move downwards

$$P=6\sin 25-friction$$
$$P=0.578N$$

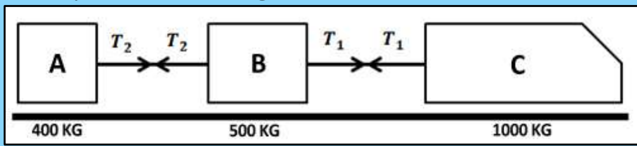
6. Connected Particles

Newton's 3rd Law of Motion:

For every action, there is an equal and opposite reaction

{Exemplar Question}

A train pulls two carriages:



The forward force of the engine is $F = 2500N$. Find the acceleration and tension in each coupling. The resistance to motion of A, B and C are 200, 150 and 90N respectively.

Solution:

To find acceleration, regard the system as a single object. The internal T's cancel out and give:

$$2500 - (200 + 150 + 90) = 1900a$$
$$\therefore a = 1.08ms^{-2}$$

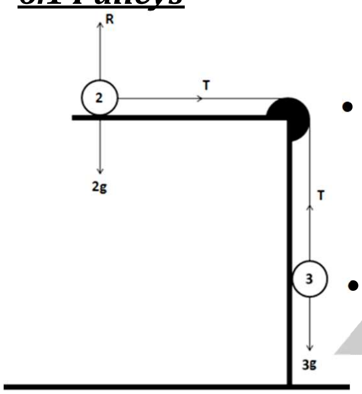
To find T_1, look at C

$$F - T_1 - 200 = 1000a$$
$$2500 - T_1 - 200 = 1000 \times 1.08$$
$$T_1 = 1220N$$

To find T_2, look at A

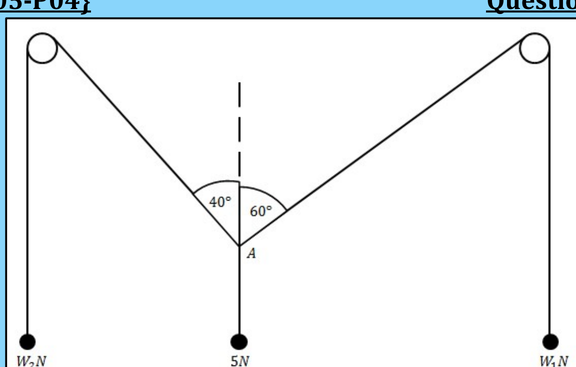
$$T_2 - 90 = 400a$$
$$T_2 - 90 = 400 \times 1.08$$
$$T_1 = 522N$$

6.1 Pulleys



- Equation 1:
  No backward force ∴
  $$T = 2a$$
- Equation 2:
  $$3g - T = 3a$$

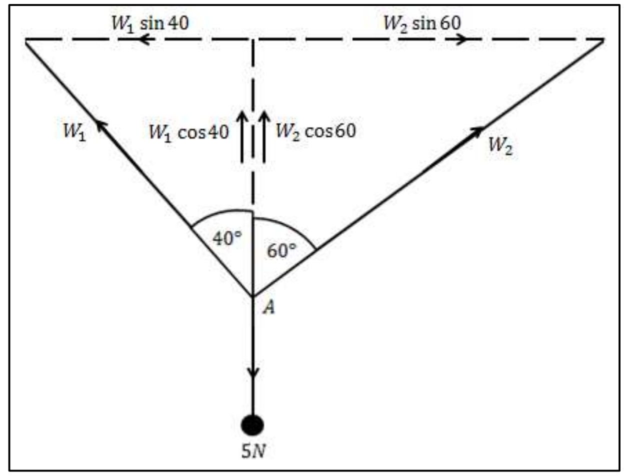
{W05-P04} **Question 3:**



The strings are in equilibrium. The pegs are smooth. All the weights are vertical. Find W_1 and W_2

Solution:

Diagram showing how to resolve forces:



Resolving forces at A vertically:

$$W_1 \cos 40 + W_2 \cos 60 = 5$$

Resolving forces at A horizontally:

$$W_1 \sin 40 = W_2 \sin 60$$

Substitute second equation into first:

$$\left(\frac{W_2 \sin 60}{\sin 40}\right) \cos 40 + W_2 \cos 60 = 5$$

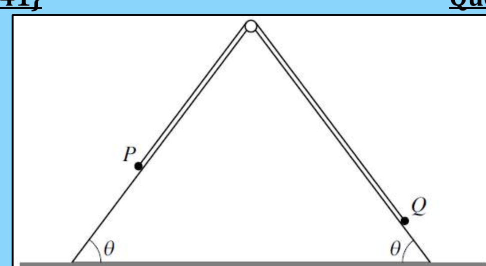
Solve to find W_2:

$$W_2 = 3.26N$$

Put this value back into first equation to find W_1

$$W_1 = 4.40N$$

{S12-P41} **Question 6:**



P has a mass of $0.6kg$ and Q has a mass of $0.4kg$. The pulley and surface of both sides are smooth. The base of triangle is horizontal. It is given that $\sin\theta = 0.8$. Initially particles are held at rest on slopes with string taut. Particles are released and move along the slope

i. Find tension in string. Find acceleration of particles while both are moving.

ii. Speed of P when it reaches the ground is $2ms^{-1}$. When P reaches the ground, it stops moving. Q continues moving up slope but does not reach the pulley. Given this, find the time when Q reaches its maximum height above ground since the instant it was released

Solution:

Part (i)

Effect of weight caused by P in direction of slope:

Effect of weight $= mg\sin\theta$ where $\sin\theta = 0.8$

Effect of weight $= 4.8N$

Effect of weight caused by Q in direction of slope:

Effect of weight $= 0.4 \times 10 \times 0.8 = 3.2N$

Body P has greater mass than body Q so when released P moves down Q moves up on their slopes $\therefore$

$$4.8 - T = 0.6a$$
$$T - 3.2 = 0.4a$$

Solve simultaneous equations:

$$\frac{4.8-T}{0.6} = \frac{T-3.2}{0.4} \quad T = 3.84N$$

Substitute back into initial equations to find a:

$$4.8 - 3.84 = 0.6a \quad a = 1.6ms^{-2}$$

Part (ii)

Use kinematics equations to find the time which it takes P to reach the ground:

$$a = \frac{v-u}{t} \text{ and } t = \frac{2-0}{1.6}$$
$$t_1 = 1.25s$$

When P reaches the ground, only force acting on Q is its own weight in the direction of slope $= 3.2N$

$$F = ma \quad -3.2 = 0.4a$$
$$a = -8ms^{-2}$$

Now calculate the time taken for Q to reach max height

This occurs when its final velocity is 0.

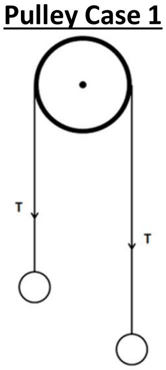
$$-8 = \frac{0-2}{t} \quad t_2 = 0.25s$$

Now do simple addition to find total time:

Total Time $= 1.25 + 0.25 = 1.5s$

6.2 Force Exerted by String on Pulley

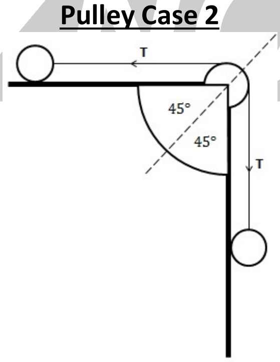
Pulley Case 1



Force on pulley = $2T$

Acts: downwards

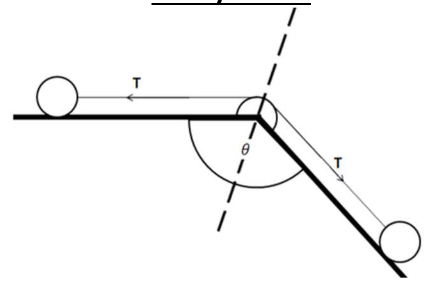
Pulley Case 2



Force on pulley = $T\sqrt{2}$

Acts: along dotted line

Pulley Case 3

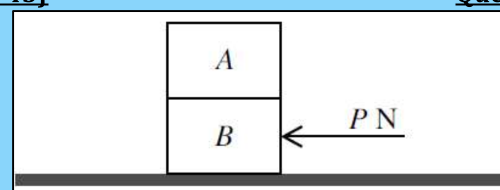


Force on pulley = $2T\cos\left(\frac{1}{2}\theta\right)$

Acts: inwards along dotted line which bisects θ

6.3 Two Particles

{S10-P43} **Question 7:**



A and B are rectangular boxes of identical sizes and are at rest on rough horizontal plane. A mass $= 200kg$ and B mass $= 250kg$. If $P \leq 3150$ boxes remain at rest. If $P > 3150$ boxes move

i. Find coefficient of friction between B and floor
ii. Coefficient of friction between boxes is 0.2. Given that $P > 3150$ and no sliding occurs between boxes. Show that the acceleration of boxes is not greater than $2ms^{-2}$
iii. Find the maximum possible value of P in the above scenario

Solution:

Part (i)

$$F = \mu N$$

F = to max P that does not move the boxes

N = to contact force of **both** boxes acting on floor

$$\therefore 3150 = \mu \times (2000 + 2500)$$
$$\mu = 0.7$$

Part (ii)

Find frictional force between A and B:

$$F = 0.2 \times 2000 \quad F = 400N$$

Use Newton's Second Law of Motion to find max acceleration for which boxes do not slide (below F)

$$400 = 200a \quad a = 2ms^{-2}$$

Part (iii)

P has to cause an acceleration of $2ms^{-2}$ on B which will pass on to A as they are connected bodies

Simply implement Newton's Second Law of Motion

$$\therefore P = (200 + 250)(2) + 3150$$

The 3150 comes from the force required to overcome the friction

$$P = 900 + 3150 \quad P = 4050N$$

7. Work, Energy and Power

Principle of Conservation of Energy:

Energy cannot be created or destroyed, it can only be changed into other forms

Work Done: $W = Fs$

Kinetic Energy: $E_k = \frac{1}{2}mv^2$

Gravitational Potential Energy: $E_P = mgh$

Power: $P = \frac{W.d}{T}$ and $P = Fv$

7.1 Changes in Energy

$$\varepsilon_f - \varepsilon_i = (Work)_{engine} - (Work)_{friction}$$

- ε_f is the final energy of the object
- ε_i is the initial energy of the object
- $(Work)_{engine}$ is the energy caused by driving force acting on the object
- $(Work)_{friction}$ is the energy used up by frictional force or any resistive force

{S05-P04} **Question 7:**

Car travelling on horizontal straight road, mass 1200kg. Power of car engine is $20kW$ and constant. Resistance to motion of car is $500N$ and constant. Car passes point A with speed $10ms^{-1}$. Car passes point B with speed $25ms^{-1}$. Car takes $30.5s$ to move from A to B.

i. Find acceleration of the car at A

ii. Find distance AB by considering work & energy

Solution:

Part (i)

Use formula for power to find the force at A

$$P = Fv$$

$$20000 = 10F \quad Driving\ force = 2000N$$

We must take into account the resistance to motion

$$\therefore F = Driving\ Force - Resistance = 2000 - 500$$

$$F = 1500$$

Use Newton's Second Law to find acceleration:

$$1500 = 1200a \qquad a = \frac{1500}{1200} = 1.25ms^{-2}$$

Part (ii)

Use power formula to find work done by engine:

$$P = \frac{w.d.}{t}$$

$$20000 = \frac{w.d.}{30.5} \quad w.d. = 610000J$$

There is change in kinetic energy of the car so that means some work done by the engine was due to this:

$$k.E.at\ A = \frac{1}{2}1200(10)^2 \quad k.E.at\ B = \frac{1}{2}1200(25)^2$$

$$Change\ in\ k.E. = k.E.at\ B - k.E.at\ A$$

$$Change\ in\ k.E. = 375000 - 60000 = 315000$$

There is also some work done against resistive force of $500N$; due to law of conservation of energy, this leads us to the main equation:

$$w.d.by\ engine = change\ in\ k.E + w.d.against\ resistance$$

$$610000 = 315000 + 500s$$

$$s = \frac{610000 - 315000}{500} = \frac{295000}{500} = 590m$$

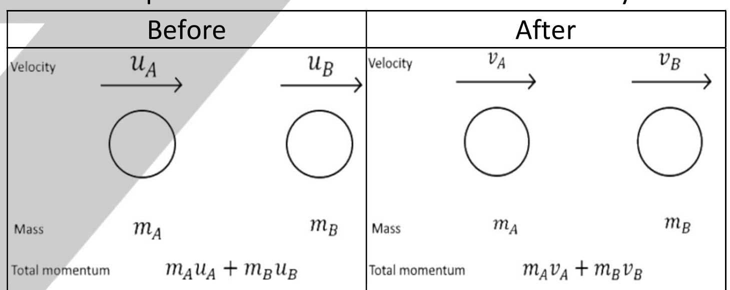
8. Momentum

- Momentum is a vector quantity, having the same direction as the velocity.
- The units of momentum are N s
- **Linear momentum:** product of mass and velocity

$$Momentum = mass \times velocity$$

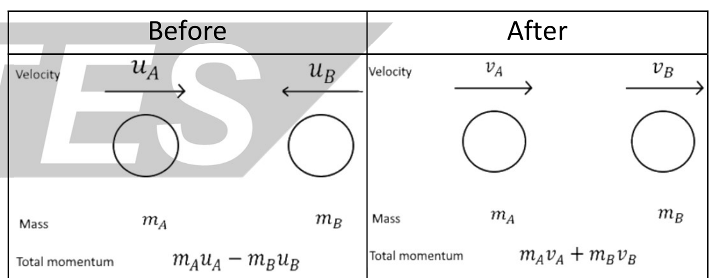
$$p = mv$$

- **Principle of conservation of linear momentum:** when bodies in a system interact, total momentum remains constant provided no external force acts on the system.



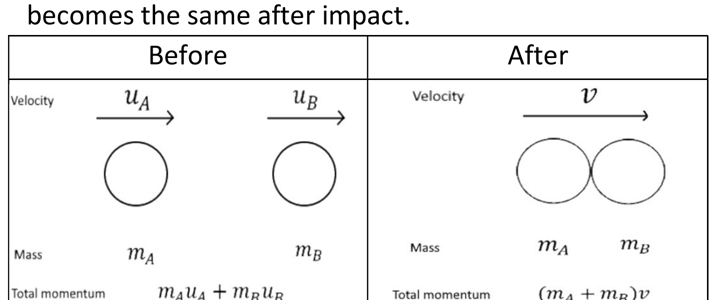
$$m_Au_A + m_Bu_B = m_Av_A + m_Bv_B$$

- When both particles A and B move towards each other



$$m_Au_A - m_Bu_B = m_Av_A + m_Bv_B$$

- When both particles stick together, their velocity becomes the same after impact.



$$m_Au_A + m_Bu_B = (m_A + m_B)v$$

{SP20-P04} **Question 3:**

Three small smooth spheres A, B and C of equal radii and of masses 4 kg, 2 kg and 3 kg respectively, lie in that order in a straight line on a smooth horizontal plane. Initially, B and C are at rest and A is moving towards B with speed 6ms^{-1}. After the collision with B, sphere A continues to move in the same direction but with speed 2ms^{-1}

i. Find the speed of B after this collision
ii. Sphere B collides with C. In this collision these two spheres coalesce to form an object D. Find the speed of D after this collision

Solution:

Part (i)

Calculate momentum of system before collision:

$$p = mv$$
$$p = (4 \times 6) + 2(0) + 3(0) = 24\ kgms^{-1}$$

Calculate momentum of system after collision:

$$p = (4 \times 2) + 2v$$

Apply conservation of momentum:

$$total\ momentum\ before = \ total\ momentum\ after$$
$$24 = 8 + 2v \qquad v = \frac{16}{2} = 8.0\ ms^{-1}$$

Part (ii)

Calculate momentum of system before collision:

$$p = (8 \times 2) + (3 \times 0) = 16\ kgms^{-1}$$

Calculate momentum of system after collision:

Note: The masses B and C combine to form D

$$p = (2 + 3)v$$
$$p = 5v$$

Apply conservation of momentum:

$$16 = 5v$$
$$v = \frac{16}{5} = 3.2\ ms^{-1}$$

9. General Motion in a Straight Line

DIFFERENTIATE →

s	v	a
displacement	velocity	acceleration

← INTEGRATE

- Particle at instantaneous rest, $v = 0$
- Maximum displacement from origin, $v = 0$
- Maximum velocity, $a = 0$

{W10-P42} **Question 7:**

Particle P travels in straight line. It passes point O with velocity $5ms^{-1}$ at time $t = 0s$.

P's velocity after leaving O given by:

$$v = 0.002t^3 - 0.12t^2 + 1.8t + 5$$

v of P is increasing when: $0 < t < T_1$ and $t > T_2$

v of P is decreasing when: $T_1 < t < T_2$

i. Find the values of T_1 and T_2 and distance OP when $t = T_2$
ii. Find v of P when $t = T_2$ and sketch velocity-time graph for the motion of P

Solution:

Part (i)

Find stationary points of v; maximum is where $t = T_1$ and minimum is where $t = T_2$

$$\frac{dv}{dt} = 0.006t^2 - 0.24t + 1.8$$

Stationary points occur where $\frac{dv}{dt} = 0$

$$\therefore 0.006t^2 - 0.24t + 1.8 = 0$$

Solve for t in simple quadratic fashion:

$$t = 30 \quad and \quad 10$$

Naturally T_1 comes before T_2

$$\therefore T_1 = 10s \quad and \quad T_2 = 30$$

Finding distance OP by integrating

$$\therefore s = \int_0^{30} v\,dt$$
$$s = \int_0^{30} (0.002t^3 - 0.12t^2 + 1.8t + 5)\,dt$$
$$s = {}_0^{30}[0.0005t^4 - 0.04t^3 + 0.9t^2 + 5t]$$
$$s = 285\ m$$

Part (ii)

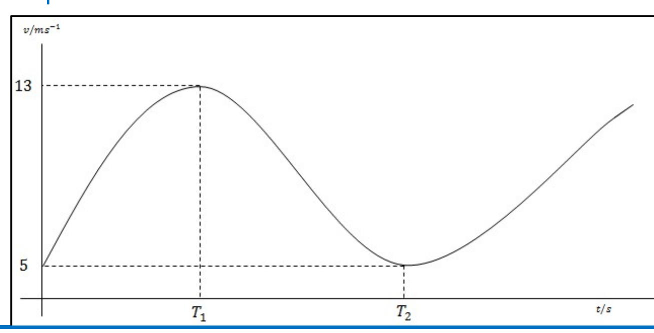
Do basic substitution to find v

$$v = 0.002t^3 - 0.12t^2 + 1.8t + 5$$
$$t = 30 \quad v = 5$$

To draw graph, find v of P at T_1 using substitution and plot roughly

$$v\ at\ T_1 = 13$$

Graph:



{S13-P42} **Question 6:**

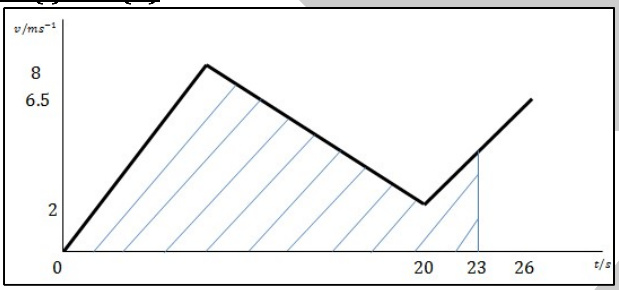
Particle P moves in a straight line. Starts at rest at point O and moves towards a point A on the line. During first 8 seconds, P's speed increases to $8ms^{-1}$ with constant acceleration. During next 12 seconds P's speed decreases to $2ms^{-1}$ with constant deceleration. P then moves with constant acceleration for 6 seconds reaching point A with speed $6.5ms^{-1}$

i. Sketch velocity-time graph for P's motion
ii. The displacement of P from O, at time t seconds after P leaves O, is s metres. Shade region of the velocity-time graph representing s for a value of t where $20 \leq t \leq 26$
iii. Show that for $20 \leq t \leq 26$,

$$s = 0.375t^2 - 13t + 202$$

Solution:

Part (i) and (ii)



Part (ii)

First find s when $t = 20$, this will produce a constant since $20 \leq t \leq 26$

$$s_1 = \frac{1}{2}(8)(8) + \frac{1}{2}(8+2)(12) = 92m$$

Finding s when $20 \leq t \leq 26$:

$$s = ut + \frac{1}{2}at^2$$

Since the distance before 20 seconds has already been taken into consideration:

$$\boldsymbol{t = t - 20}$$

$$a = \frac{6.5 - 2}{6}$$

$$a = 0.75$$

$$\therefore s_2 = 2(t-20) + \frac{1}{2}(0.75)(t-20)^2$$

$$s_2 = 2t - 40 + 150 + 0.375t^2 - 15t$$

$$s_2 = 0.375t^2 - 13t + 110$$

Finally add both to give you s

$$s = s_1 + s_2$$

$$s = 0.375t^2 - 13t + 110 + 92$$

$$s = 0.375t^2 - 13t + 202$$

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